

Buy-Out Prices in Online Auctions: Multi-Unit Demand*

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Abstract

On many online auction sites it is now possible for a seller to augment his auction with a maximum or buy-out price. The use of this instrument has been justified in “one-shot” auctions by appeal to impatience or risk aversion. Here we offer additional justification by observing that trading on internet auctions is not of a “one-shot” nature, but that market participants expect more transactions in the future. This has important implications when bidders desire multiple objects. Specifically, it is shown that an early seller has an incentive to introduce a buy-out price, if similar products are offered later on by other sellers. The buy-out price will increase revenue in the current auction, but revenue in future auctions will decrease, as will the sum of revenues. In contrast, if a single seller owns multiple units, overall revenue will increase, if buyers anticipate the use of buy-out prices in the future by this seller. In both cases, an optimally chosen buy-out price introduces potential inefficiencies in the allocation.

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1 Introduction

The presence of buy-out prices¹ in online auctions has thus far been explained by focusing on a *single* auction and assuming that individuals exhibit either *risk aversion* or *impatience*.² In this paper we take a somewhat broader view of auction markets, realizing, in particular, that buyers and sellers alike are aware of the fact that new products will be offered on the market in the future. This will tend to depress revenue in today's auctions, as buyers know that close substitutes will be offered tomorrow. In this *dynamic* environment we will show that there are at least two reasons to introduce buy-out prices, even if agents are patient and risk neutral.³

Buy-out prices or maximum prices in online auctions were noted by Lucking-Reiley (2000) in his empirical overview of auction activities on the Internet. Since (sell) auctions are ostensibly staged to illicit high prices in situations where markets are thin and sellers are short on information about the willingness-to-pay of potential buyers, such buy-out prices may appear surprising. In fact, Lucking-Reiley explicitly posed this as a challenge to theorists. In addition, he quoted evidence to suggest that the *exercise* of posted buy-out options is not uncommon in online auctions.⁴

Reynolds and Wooders (2003) provide some additional information on the *frequency of buy-out prices* in *Yahoo!* and *eBay* auctions, though, *not* on the frequency with which the option was *exercised* by some bidder. The categories sampled on March 27, 2002, were automobiles, clothing, DVD players, VCR's, digital cameras and TV sets. A total of 1.248 auctioned items were sampled from *Yahoo!*, of which 842 had a buy-out price posted by the

¹Alternatively, this is often referred to as *buy prices* or *maximum prices*. In offline settings, this phenomenon also has a certain affinity with "\$*xx* or best offer", where it is, presumably, implicit that, if someone makes an offer of \$*xx*, then the trade is finalized immediately, while if someone makes a lower offer initially, then the seller will wait a while to see if a better offer comes along. Also, a buy-out price has a certain similarity with a massive *jump bid* intended to end an auction quickly.

²See, Budish and Takeyama (2001), Mathews (2002), Reynolds and Wooders (2003) and Hidvégi, Wang and Whinston (2003).

³Throughout this paper potential buyers bid *non-cooperatively*. In future work we hope to return to the use of buy-out prices in auctions where sellers try to respond to possible *bidder collusion*.

⁴He quotes the case of *LabX* (a lab equipment auction site), where buy-out options are exercised by some bidder in 10% of the cases where they appear. Hence, buy-out prices do more than just attract attention.

seller (roughly, 66%). In similar fashion, 31.142 auctioned items were sampled from *eBay*, of which 12.480 had a buy-out price posted by the seller (roughly, 40%). There is some variation across the categories of goods sampled, but the frequency of buy-out prices never drops below 25% in the sample. Hence, in these categories, at least, the appearance of buy-out prices is very frequent.

For *eBay*, Mathews (2002) presents some numbers on the *frequency with which buy-out options are exercised* when offered.⁵ For two categories of games (racing and sports) for *Sony PS2*, Mathews reports that on January 29 - 30, 2001, 210 items were on offer. A buy-out option was available on 124 items (59%), and it was exercised 34 times (27% of the times it was offered). So, at least in these categories, the exercise frequency is high.

Formally, we analyze *eBay*'s version of a buy-out price, termed the *Buy It Now* price. Here is how the *Buy It Now* price roughly works from the seller's viewpoint:⁶ "If a buyer is willing to meet your *Buy It Now* price before the first bid comes in, your item sells instantly and your auction ends. Or, if a bid comes in first, the *Buy It Now* option disappears. Then your auction proceeds normally." Hence, in *eBay* auctions, the buy-out price is temporary.⁷

Throughout this paper we assume that potential buyers or bidders have *multi-unit demands*, with diminishing marginal utility. With two objects for sale and at least two bidders, it has been shown by Black and de Meza (1992) that auction revenue will increase over time and that the auction outcome is efficient under these assumptions. In particular, in a sequence of second-price or English auctions, the seller offering his good today will not earn as much as a competing seller offering a similar good tomorrow, that is, prices are increasing.⁸

⁵He also presents aggregate numbers on the frequency with which buy-out prices are offered at *eBay*. The reported range around 40% is roughly in line with the numbers reported for specific categories by Reynolds and Wooders (2003).

⁶For more details on the *eBay* version and other versions of a buy-out price, see e.g. Lucking-Reiley (2000), Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2003).

⁷For more details on the *Buy It Now* feature in *eBay* auctions the reader should consult pages.ebay.com/help/sell/bin.html. *eBay* introduced this feature in January 2001.

⁸In fact, Black and de Meza (1992) were interested in what some have referred to as *The Declining Price Anomaly*. Therefore, they went on to consider an option of the following kind: the winner of the first item is given the option of buying the second item at the same price. This, apparently, is observed in certain multi-unit auctions, and it is enough to lead to a declining price path.

However, for the case with *two individual sellers*, we show that the first seller can always increase his revenue by introducing a buy-out price. The revenue to the second seller is adversely affected, as is overall revenue. An optimally chosen buy-out price in the first auction also introduces *inefficiency*, in the sense that *a bidder who should have won no object wins one*. Our analysis is partial in the following sense. We consider a sequence of two second-price (or English) auctions, allowing the first seller the possibility of introducing a buy-out price without giving the second seller the opportunity to respond in kind. Thus, we essentially show that an auction market without buy-out prices is unstable, in the sense that current sellers will try to force the auction site to (at least temporarily) allow buy-out prices.

Next, we consider the consequences of buy-out prices for a *single seller* intending to sell two objects. We show that this seller can increase his total expected revenue by augmenting the second auction with a buy-out price, which depends on the outcome of the first auction. The buy-out price should be set fairly low, thus allowing the winner of the first auction a disproportionately large chance of winning the second auction as well. Hence, the sequence of auctions is *inefficient*, in the sense that *one buyer may win two objects when efficiency dictates he should only win one*. In this case overall revenue will increase. The reason is the same as that which induces a monopolist to offer quantity discounts that are detrimental to efficiency: buyers with high demand contribute with higher marginal revenue on two objects than buyers with low demand do on one object.

The rest of the paper is organized as follows. In Section 2 we set up a simple model and present the results for the bench-mark case where a sequence of two second-price auctions is staged. Then, Section 3 shows that the first seller among a pack of competing sellers can increase his lot by offering a buy-out price. In Section 4 we comment further on the relationship between buy-out prices, inefficiencies, revenue non-equivalence and “ironing” of marginal revenues. This section also explains the special cases where bidders have unit demands and “flat”, multi-unit demands. Section 5 examines the use of buy-out prices by a single seller offering more than one unit. Section 6 contains a few concluding remarks. A selection of proofs is in the Appendix.

2 Model and Bench-Mark

In this section we first set up the model and then derive results for the bench-mark case where a sequence of two second-price auctions is staged.

2.1 Model

We assume that two objects are offered for sale sequentially,⁹ and that there are two potential buyers on the market. Hence, the number of objects coincide with the number of buyers, this number being equal to two in order to make the analysis manageable. Each buyer i , $i = 1, 2$, is characterized by a type, v_i , drawn from a continuously differentiable distribution function, $F(v_i)$, without mass points. We assume that $v_i \in [\underline{v}, \bar{v}]$. The value to bidder i of the first unit purchased is v_i , while the value of the second unit is kv_i , $0 < k < 1$. Hence, each bidder desires both units, but individual demands are downward sloping.

2.2 Two straight second-price auctions

To keep the analysis simple, we ignore the use of reserve prices in the following.¹⁰ In this setting, Black and de Meza (1992) were the first¹¹ to solve for equilibrium strategies in a sequence of two second-price (or English) auctions, under more general assumptions than those considered here.¹² Applied to our set of assumptions, they find the following.

⁹The two objects are considered homogenous by the bidders, or they are simply two units of the same good.

¹⁰Reserve prices are generally useful because they allow sellers to ration output by excluding potential buyers with low valuations. Reserve prices, thus, affect the probability that objects are sold. The effect of buy-out prices is different. In particular, buy-out prices do not influence the probability that objects are sold, but they may change the identity of the winners. It follows that a buy-out price is not a substitute for a reserve price, and that it may have a role to play, even when a reserve price is present.

¹¹See also Katzman (1999).

¹²Black and de Meza explicitly consider sealed-bid auctions, while they also have an informal discussion of English auctions. Throughout our formal analysis, we restrict attention to a setting with two bidders, in which case second-price and English auctions are equivalent. With more than two bidders this equivalence may break down. In the informal discussion immediately below, we comment on some key properties of second-price, sealed-bid auctions with arbitrary numbers of bidders.

Proposition 1 (Black and de Meza (1992)) *When there are two a priori symmetric agents in the game, the unique symmetric equilibrium is for agent i to bid kv_i in stage one, and bid v_i in stage two if stage one was lost, and kv_i otherwise. The equilibrium outcome is efficient.*

Thus, in the last round, a bidder simply bids his valuation of the remaining object. This, however, depends on whether the bidder won or lost the first object. In the first round, each bidder bids k times his valuation for the first item won. Hence, the first object is sold for a price equal to k times the lowest valuation (that is, $k \cdot \min\{v_1, v_2\} = \min\{kv_1, kv_2\}$), while the second object is sold for a price equal to the minimum of k times the highest valuation and the lowest valuation (that is, $\min\{k \cdot \max\{v_1, v_2\}, \min\{v_1, v_2\}\} = \min\{\max\{kv_1, kv_2\}, \min\{v_1, v_2\}\}$). From this, it follows that the revenue of the first auction is lower than the revenue of the second (that is, $\min\{kv_1, kv_2\} < \min\{\max\{kv_1, kv_2\}, \min\{v_1, v_2\}\}$).¹³

To see what is going on here, let us start by making a few general remarks on second-price, sealed-bid auctions in the independent, private values case with n bidders. We first note that in case of *symmetric, increasing* bidding strategies, the fine details of any bidder's bid function are only consequential if there happens to be a competing bidder who has a valuation very close to that of the bidder in question. Hence, in equilibrium a bidder's strategy is pinned down by an indifference relation: the bidder should be indifferent between winning and losing, if his toughest competitor is identical to himself. To proceed, let us take the perspective of bidder i and label his rivals j , $j = 1, 2, \dots, n - 1$. Now, i 's competitors have random valuations of the first item denoted Y_j with associated order-statistics $Y_{[1]} \geq Y_{[2]} \geq \dots \geq Y_{[n-1]}$. Let i be male and all the rivals female.

In a *one-shot*, second-price auction bidder i essentially bids what he expects it to take to win the item, *if* he is the "top dog" - the high-valuation bidder - *and* there is someone like him among the rivals. The relevant indifference relation can be written as

$$\overbrace{v_i - b(E(Y_{[1]} | Y_{[1]} = v_i))}^{\text{just winning}} = \overbrace{0}^{\text{just losing}}$$

However, $E(Y_{[1]} | Y_{[1]} = v_i) = v_i$, and the optimal bid of i is given by

$$b(v_i) = E(Y_{[1]} | Y_{[1]} = v_i) = v_i$$

¹³Assume, without loss of generality, that $v_1 \geq v_2$. Then, first-auction revenue, kv_2 , is clearly less than second-auction revenue, $\min\{kv_1, v_2\}$.

Thus, we obtain the familiar result that it is optimal for bidder i to bid his valuation.

In a *sequence of two* second-price auctions things are a little more complicated. Consider the last round first. If i won the first item, his valuation of the second item is $v_i^2 = kv_i$. Then, in the last round, bidder i 's indifference relation is predicated on $Y_{[1]} = kv_i$ (the toughest competitor is like him *at this stage*). Thus, we can write

$$\overbrace{v_i^2 - b^2(E(Y_{[1]} | Y_{[1]} = v_i^2))}^{\text{just winning second}} = \underbrace{0}_{\text{just losing second}}$$

where $b^2(\cdot)$ denotes the second-round bid. Substituting for v_i^2 and noting that $E(Y_{[1]} | Y_{[1]} = kv_i) = kv_i$, we obtain

$$b^2(v_i^2) = b^2(kv_i) = E(Y_{[1]} | Y_{[1]} = kv_i) = kv_i$$

Similarly, if i lost the first item, his valuation of the second item is $v_i^2 = v_i$. Then, in the last round, bidder i 's indifference relation is predicated on $\max\{kY_{[1]}, Y_{[2]}\} = v_i$ (the toughest competitor is like him *at this stage*). We can write this as

$$\overbrace{v_i^2 - b^2(E(\max\{kY_{[1]}, Y_{[2]}\} | \max\{kY_{[1]}, Y_{[2]}\} = v_i^2))}^{\text{just winning second}} = \underbrace{0}_{\text{just losing second}}$$

and we obtain

$$b^2(v_i^2) = b^2(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} | \max\{kY_{[1]}, Y_{[2]}\} = v_i) = v_i$$

The upshot is that bidder i should bid kv_i in the last round if he won the first and v_i if he lost. This is just bidding one's value in the last round.

More interestingly, consider the first round. We note that if i is the "top dog" and there is someone like i in the pack of rivals, then they each win one item in equilibrium.¹⁴ Hence, optimal bidding by i in the first round is

¹⁴When strategies are symmetric and increasing, the first auction is won if the toughest rival has a lower valuation, and lost if the toughest rival has a higher valuation. If the toughest rival has the same valuation as the agent himself, there is a tie, and the winner of the first auction is determined by chance. We argue that the agent must be indifferent between winning and losing the first auction in this case. Assume, to the contrary, that the agent prefers to win (lose) against an identical, strongest rival. Then, the agent should bid more (less) aggressively at the outset to win (lose) with probability one (rather than one half). This implies that the original strategies are not in equilibrium, unless the indifference condition is satisfied.

derived from an indifference between winning the first and the second item, which (using the results already derived) we can write as

$$\overbrace{[v_i - b^1(v_i)] + 0}^{\text{just winning first and losing second}} = \overbrace{0 + [v_i - E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i)]}^{\text{just losing first and winning second}}$$

$$b^1(Y_{[1]}) \text{ with } Y_{[1]} = v_i$$

Thus, in the first auction, bidder i should bid what he expects to have to pay to win the second, if he just loses the first. That is, optimal bidding in the first round is captured by

$$b^1(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i) = E(\max\{kv_i, Y_{[2]}\} \mid Y_{[1]} = v_i)$$

In the general case with n bidders, we conclude that bidder i should bid the expectation of the *maximum* of k times his strongest rival's valuation of the first item and his second strongest rival's valuation of the first item *predicated on the strongest rival being identical to himself*.

Essentially, the reasons why the expected revenue in the second auction is higher than in the first are as follows. Since auctions are both “second price”, their prices (hence, revenues) are determined by the runners-up, that is, the bidders with the second-highest marginal valuations. Furthermore, any bidder bases his bid in the first auction on the assumption that his strongest rival has the same valuation. Note that if the runner-up and the winner of the first auction indeed have the same valuations, expected prices (revenues) will be constant, which is just another way of stating the indifference condition. However, the probability that the valuations of two bidders coincide is zero. In a sense, the runner-up of the first auction underestimates the valuation of the winner or the price in the second auction. Hence, expected prices (revenues) are increasing.¹⁵

Finally, let us specialize to the two-bidder case. When $n = 2$, $Y_{[2]}$ is zero by construction, and the optimal bid of i reduces to

$$b^1(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i) = E(\max\{kY_{[1]}, 0\} \mid Y_{[1]} = v_i) = kv_i$$

as stated in the proposition above.

Our next result characterizes the expected revenues associated with the equilibrium strategies in Proposition 1.

¹⁵Observe that if $k = 0$ (unit demands), the fact that the runner-up underestimated the valuation of the winner is irrelevant, because the winner does not compete in the second auction.

Lemma 1 *In two straight second-price auctions with two bidders, the expected revenues in the first and second auctions are, respectively,*

$$ER_1^{SSP} = k \int_{\underline{v}}^{\bar{v}} 2x(1 - F(x))f(x)dx \quad (1)$$

and

$$\begin{aligned} ER_2^{SSP} = & \int_{\underline{v}}^{\max\{\underline{v}, k\bar{v}\}} 2x(1 - F(\frac{x}{k}))f(x)dx \\ & + k \int_{\underline{v}}^{\bar{v}} 2x(F(x) - F(\max\{\underline{v}, kx\}))f(x)dx \end{aligned} \quad (2)$$

Proof. In the first auction, players bid k times their valuation, and the price is equal to the lowest bid. Hence, expected revenue is k times the expected value of the second highest valuation, which is just (1).

In the second auction there are two possible outcomes, depending on whether the same or different bidders win the two objects. The first term in (2) captures the possibility that the winner of the first object is also the winner of the second. Since the loser of the first auction bids his valuation, x say, and the winner bids k times her valuation, the price is precisely x when one player has valuation x and the other has a valuation that exceeds x/k .

However, it is also possible that the runner up in the first auction becomes the winner of the second, and this is the second term in (2). If the winner of the first auction has valuation x , her bid will be kx in the second auction. Hence, the price is kx in the second auction when one agent has type x , and the other agent has a type that is lower than x , yet sufficiently high that the bid submitted by this player exceeds kx . ■

From this we note that $ER_1^{SSP} \rightarrow 0$ and $ER_2^{SSP} \rightarrow 0$ as $k \rightarrow 0$. This, however, is just a special version of Weber's (1983) result that a sequence of second-price (or English) auctions where bidders have *unit demands* yields the same expected revenue to all sellers. With only two bidders and two items for sale, the equilibrium revenue is zero to both sellers. It is impossible to extract rent from buyers when there is no excess demand, recalling our assumption of no reserve prices.¹⁶

¹⁶Our general argument above for the n bidder case captures further aspects of Weber's results. With $k = 0$ (unit demands) bidding in both the first and the second auction

Similarly, we note that $ER_1^{SSP} \rightarrow \int_{\underline{v}}^{\bar{v}} 2x(1 - F(x))f(x)dx = E(v_{[2]})$ and $ER_2^{SSP} \rightarrow \int_{\underline{v}}^{\bar{v}} 2x(1 - F(x))f(x)dx = E(v_{[2]})$ as $k \rightarrow 1$. $E(v_{[2]})$ is just the expectation of the lowest of the two independent random draws from $F(v)$. When $k = 1$, *individual demands are horizontal*, and the behavior in the second auction is independent of the outcome of the first auction. The high valuation bidder will win both objects at a price of $v_{[2]}$, and revenue is the same in both auctions.

Given the increasing path of revenues over two straight second-price auctions, it is clear that the first of *two independent sellers* has an incentive to change the auction format.¹⁷ In this paper we shall *first* restrict attention to the possible role of a buy-out price in the first auction when two independent sellers are selling identical objects. The first seller is interested in shifting revenues from the second to the first auction, while we shall also be interested in the consequences for efficiency and total revenue when the buy-out price is set optimally by the first seller. *Subsequently*, we turn to the case where there is a *single seller* who attempts to sell two identical objects in a sequence of auctions. Absent discounting (impatience), this seller is only interested in total expected revenue from the two auctions, while he is indifferent as to whether revenues are increasing or decreasing over the sequence. We show, however, that a suitably chosen buy-out price in the second auction, depending on the outcome of the first auction, can increase the total expected revenue of a single seller at the potential expense of efficiency.

To ease the exposition, we make the following assumption in the remainder of the paper.

Assumption 1. $k\bar{v} > \underline{v}$

Essentially, this means that *a priori* there is uncertainty as to whether an efficient mechanism would allocate both objects to the same buyer or one object to each potential buyer. Hence, it is entirely possible that bidder i 's valuation of a second object exceeds bidder j 's valuation of the first object,

is ultimately based purely on the expected *second* highest value among a bidder's rivals, thus, on the *third order* statistic $v_{[3]}$ of the n random valuations. From this it follows that expected revenue is the same in the two auctions when $k = 0$ (cf. the observation in the previous footnote).

¹⁷That is, short of moving to the last spot if possible. If selling-time is an endogenous variable, the two symmetric sellers might conceivably be involved in a war of attrition to become the last seller. This, however, is not the topic of this paper, and seller positions in the auction sequence are assumed to be exogenous.

$kv_i > v_j$. Economically, this is the most interesting and challenging case. We could alternatively refer to this as the case with *overlap*. In the alternative, *non-overlap* case, $k\bar{v} < \underline{v}$, any efficient mechanism would allocate one object to each potential buyer. In this case, a bidder who has already won one object ceases to be an effective competitor for the second.¹⁸

Given Assumption 1, we note that (2) can be written as

$$\begin{aligned}
ER_2^{SSP} &= \int_{\underline{v}}^{k\bar{v}} 2x(1 - F(\frac{x}{k}))f(x)dx + k \int_{\underline{v}}^{\frac{\underline{v}}{k}} 2xF(x)f(x)dx \\
&\quad + k \int_{\frac{\underline{v}}{k}}^{\bar{v}} 2x(F(x) - F(kx))f(x)dx
\end{aligned} \tag{3}$$

Below, two types of inefficiency will be identified. First, a mechanism may allocate one object to a bidder who would have received no object in an efficient mechanism. As we shall see this will be a feature of the mechanism for the case with two independent sellers where the first seller sets an optimal buy-out price. Likewise, a mechanism may allocate both objects to a bidder who would only have received one object in an efficient mechanism. This will arise in the case where a single seller sets a buy-out price in the second auction which depends on the outcome of the first auction.

2.3 Example

To add some further insights into the results above, let us consider the *uniform* case with $v \in [0, 1]$, that is, $\underline{v} = 0$, $\bar{v} = 1$, $f(v) = 1$ and $F(v) = v$. Note that Assumption 1 (*overlap*) is satisfied in this example.

The expected revenues in the two auctions reduce to

$$ER_1^{SSP} = k \int_0^1 2x(1 - x)dx = \frac{1}{3}k$$

¹⁸Thus, Assumption 1 is pretty innocuous. However, it allows us to economize on notation in the formal analysis below. For completeness, we have included Appendix B, which shows that all the results in Section 3 below hold with minor modifications when Assumption 1 is not met. The interested reader should consult Appendix B when the results in Section 3 have been derived.

and

$$\begin{aligned} ER_2^{SSP} &= \int_0^k 2x\left(1 - \frac{x}{k}\right)dx + k \int_0^1 2x(x - kx)dx \\ &= \frac{1}{3}k + \frac{1}{3}k(1 - k) = ER_1^{SSP} + \frac{1}{3}k(1 - k) \end{aligned}$$

We plot these expected revenues against k in Fig. 1, where ER_2^{SSP} is the *heavy* line, while ER_1^{SSP} is *thin*.

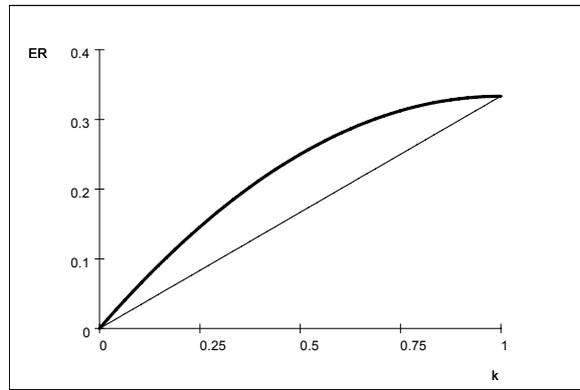


Fig. 1: Two straight second-price auctions

The *ratio* between expected revenues in the first and second auction, $RR(SSP) = \frac{ER_1^{SSP}}{ER_2^{SSP}} = \frac{1}{2-k}$, is illustrated in Fig. 2. Note the discontinuity at $k = 0$. When $k = 0$, both sellers earn nothing, that is, the same. However, when k is small, but strictly positive, we observe that the winner of the first auction is very unlikely also to be the winner of the second auction. Hence, the expected revenue in the first auction is k times (the expected value of) the second highest valuation, while the expected revenue in the second auction is *approximately* k times (the expected value of) the highest valuation. For the uniform case considered here, the ratio between the expected value of the highest ($2/3$) and the expected value of the second highest valuation ($1/3$) is exactly $1/2$.

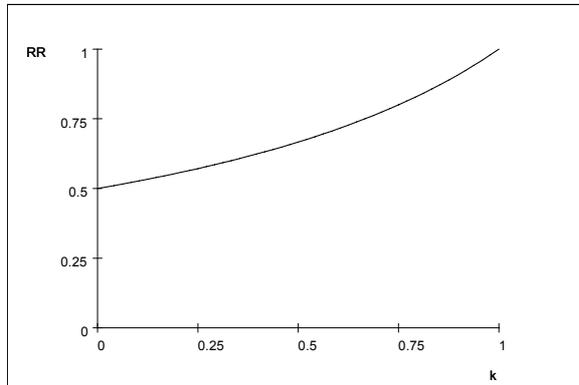


Fig. 2: The revenue-ratio in two SSP auctions

From this example it is immediate that the difference in expected revenues is significant unless k is close to one (demands are near-horizontal). For example, if $k = \frac{1}{3}$, then $ER_1^{SSP} = \frac{1}{9} \approx 0.11$ and $ER_2^{SSP} = \frac{5}{27} \approx 0.19$, and it follows that the (expected) first-auction revenue is only *60%* of the second-auction revenue.

3 Competing Sellers

We now turn to the case where two different sellers each own one object initially. We assume that the two objects are offered sequentially, and that there are two potential buyers on the market. We allow the first seller to stipulate a buy-out price of the *eBay*-variety (*Buy It Now*) and, thus, consider the following augmented game:

- 1 Seller 1 announces a buy-out price, B . At this stage bidders can submit a bid of B or refrain from bidding. The object is sold at the price B if at least one bidder bids B . If both bidders bid B , one bidder is picked at random to win. If no one bids B , a normal second-price auction is staged. The price can exceed B in this event.
- 2 Seller 2 auctions off the second item, using a second-price auction.

Thus, in stage 1 of this game, the bidders *first* have to decide whether to take the buy-out price B or leave it. If one or more bidders take the buy-out price, the first auction ends, and the winner pays B . If no one takes the

buy-out price, the first stage continues to a standard second-price auction. The second stage simply consists of a standard second-price auction.

We first derive the relationship between the level of B and the valuations of bidders who will take this buy-out price. Then we look at the relationship between the buy-out price and the expected revenues to the two sellers, including how they are ranked. Finally, we determine the optimal buy-out price for the first seller. Recall that Assumption 1 is assumed to hold throughout.

3.1 General results

We will look for a symmetric equilibrium in this augmented game in which bidders with valuations above some level \hat{v} take the buy-out price B in stage 1, while bidders with valuations below \hat{v} do not. In the augmented game, it is clear that if no bidder takes B , then it is common knowledge in equilibrium that both bidders have a type below \hat{v} . That is, beliefs are symmetric, and the logic of Proposition 1 (Black and de Meza (1992)) applies to the remainder of stage 1 and to stage 2. Hence, in stage 1 bids will be kv_i , where $v_i < \hat{v}$, $i = 1, 2$. Further, regardless of how the good is sold in stage 1, it is well known that the bid in stage 2 will be kv_i if bidder i won the first auction, and v_i otherwise. In the following we suppress the subscript when this can be done without confusion.

In the equilibrium of the augmented game, a given value of B will induce a set $[\hat{v}, \bar{v}]$ of bidder types to take the buy-out price B in stage 1. Changing B will change \hat{v} . Hence, we can determine which \hat{v} to target, and chose B accordingly. Thus, we write $B(\hat{v})$ as the value of B that induces bidder types above \hat{v} to take B in a symmetric equilibrium. This allows us to state the following result.

Proposition 2 *Let $m(\hat{v}) = \min\{\bar{v}, \frac{\hat{v}}{k}\}$, and let $B(\hat{v})$ be defined by*

$$B(\hat{v})(1 + F(\hat{v})) = \hat{v}(1 - F(m(\hat{v}))) + \int_{\underline{v}}^{m(\hat{v})} kxf(x)dx + \int_{\underline{v}}^{\hat{v}} kxf(x)dx \quad (4)$$

Then, it is an equilibrium for bidders with $v \in [\hat{v}, \bar{v}]$ to bid $B(\hat{v})$ in stage 1 and for bidders with $v \in [\underline{v}, \hat{v})$ not to.

Proof. See Appendix A. ■

It is easily seen that $B(\bar{v}) = kE(v)$. In addition, $B(\cdot)$ may not be monotonic, implying that for a given value of B , there could be multiple

symmetric equilibria. However, as shown below, for any distribution and $k \in (0, 1)$, the first seller can strictly increase his revenue by offering a buy-out price that will be accepted with positive probability. First, though, we can state the following result on the expected revenues in the two stages given some buy-out price, $B(\hat{v})$.

Proposition 3 *The expected revenue in the first auction is*

$$ER_1(\hat{v}) = k(1 - F(\hat{v})) \left(\frac{\hat{v}}{k}(1 - F(m(\hat{v}))) + \int_{\hat{v}}^{m(\hat{v})} xf(x)dx \right) + k \int_{\underline{v}}^{\hat{v}} 2x(1 - F(x))f(x)dx \quad (5)$$

while the expected revenue in the second auction is

$$ER_2(\hat{v}) = \int_{\underline{v}}^{km(\hat{v})} 2x(1 - F(\frac{x}{k}))f(x)dx + \int_{km(\hat{v})}^{k\bar{v}} x(1 - F(\frac{x}{k}))f(x)dx + k \int_{\underline{v}}^{\hat{v}} 2x(F(x) - F(\max\{\underline{v}, kx\}))f(x)dx + k \int_{\hat{v}}^{m(\hat{v})} 2x(F(\hat{v}) - F(\max\{\underline{v}, kx\}))f(x)dx + k \int_{\hat{v}}^{m(\hat{v})} x(1 - F(\hat{v}))f(x)dx + k \int_{m(\hat{v})}^{\bar{v}} x(1 - F(kx))f(x)dx \quad (6)$$

Proof. For (5) see below, and for (6) see below and Appendix A. ■

We sketch the main arguments. First, consider the expected revenues in the *first auction*. When at least one of the bidders has a valuation of at least \hat{v} , the buy-out price is taken and the first seller receives $B(\hat{v})$. This event has a probability $1 - F^2(\hat{v})$. In contrast, if both bidders have valuations less than \hat{v} (i.e., $\max\{v_i, v_j\} < \hat{v}$), the buy-out price is not taken, and the first stage continues to a second-price auction where each bidder bids kv_i according to Proposition 1. Thus, the first seller receives k times $\min\{v_i, v_j\}$. This event has a probability $F^2(\hat{v})$. We conclude that the expected revenue to the first seller given $B(\hat{v})$ can be written as

$$ER_1(\hat{v}) = (1 - F^2(\hat{v})) \times B(\hat{v}) + F^2(\hat{v}) \times kE(\min\{v_i, v_j\} \mid \max\{v_i, v_j\} < \hat{v})$$

However, $E(\min\{v_i, v_j\} \mid \max\{v_i, v_j\} < \hat{v})$ is just the expected value of the second-order statistic, $v_{[2]}$, given that the first-order statistic, $v_{[1]}$, is less than \hat{v} . Denote the density of $v_{[2]}$ given $v_{[1]} < \hat{v}$ by $h^*(v)$. Then $h^*(v) = \frac{2f(v)(F(\hat{v})-F(v))}{F^2(\hat{v})}$ and we can write

$$\begin{aligned} E(\min\{v_i, v_j\} \mid \max\{v_i, v_j\} < \hat{v}) &= \int_{\underline{v}}^{\hat{v}} v h^*(v) dv \\ &= \frac{1}{F^2(\hat{v})} \int_{\underline{v}}^{\hat{v}} 2v(F(\hat{v}) - F(v))f(v)dv \end{aligned}$$

Hence, expected revenue in the first auction given $B(\hat{v})$ (or simply \hat{v}) can be written as

$$\begin{aligned} ER_1(\hat{v}) &= (1 - F^2(\hat{v})) \times B(\hat{v}) + k \int_{\underline{v}}^{\hat{v}} 2v(F(\hat{v}) - F(v))f(v)dv \\ &= [B(\hat{v})(1 + F(\hat{v}))](1 - F(\hat{v})) + k \int_{\underline{v}}^{\hat{v}} 2v(F(\hat{v}) - F(v))f(v)dv \end{aligned}$$

Inserting $B(\hat{v})(1 + F(\hat{v}))$ from Proposition 1, we can write this as (5).

The derivation of the expected revenue in the *second auction* is slightly more complicated, and we relegate the formal derivation of (6) to Appendix A. However, in the second auction, the object will be bought either by the winner of the first auction, or by the loser.

The first and second term in (6) capture revenue in the former case. Assuming that the loser of stage 1 has valuation x , and bids x in stage 2, he will lose the second auction if the other bidder's bid exceeds x , which requires that the rival has a valuation which is at least x/k . The *first* term in (6) then accounts for the possibility that one bidder has a valuation below \hat{v} (and thus does not accept B) and also below \bar{v}/k (implying the existence of a bidder type which has a higher marginal valuation of both units), and that the other bidder has a very high valuation, allowing him to win both auctions. The *second* term in (6) describes the case where both bidders accepted B , but that the (random) loser has a valuation which is low relative to the winner. This exhausts the possibilities that the winner is the same in both stages.

The remaining terms in (6) are relevant if the winner of stage 1 loses stage 2. Assuming this bidder has a valuation of x , say, the price in the second auction will then be equal to the bid from this bidder, namely kx . The *third* term in (6) is for cases where the winner of stage 1 did not accept the buy-out

price, and where the other bidder (who must have a lower valuation) submits a bid higher than kx in stage 2. The *fourth* term in (6) is relevant when the winner of the first auction bid B , but was the only one to do so. Furthermore, the *fifth* term in (6) is for cases where both bidders bid B , but where the (random) winner of stage 1 loses stage 2 because his valuation is so small that he is certain to lose stage two given the fact that the other bidder has a valuation higher than \hat{v} . Finally, the *sixth* term in (6) applies when both bidders bid B , and the (random) winner of stage 1 has a valuation which is low relative to the loser, allowing the latter to win stage 2. This exhausts the possibilities that the loser of stage 1 wins stage 2.

To end this subsection we can state two more general results. The first establishes monotonicity of second auction revenues and total revenues in the cut-off valuation, while the second and main result establishes the revenue ranking.

Lemma 2 (Monotonicity) (i) $ER_2(\hat{v})$ is strictly increasing for $\hat{v} \in [\underline{v}, \bar{v}]$.
(ii) $ER_1(\hat{v}) + ER_2(\hat{v})$ is strictly increasing for $\hat{v} \in [\underline{v}, k\bar{v})$, and constant for $\hat{v} \in [k\bar{v}, \bar{v}]$.

Proof. See Appendix A. ■

The fact that $ER_2(\hat{v})$ is increasing can easily be understood by the following two observations. First, if the first auction is won by the bidder with the lowest valuation (because both bidders bid the buy-out price B , and the low-valuation bidder is randomly picked as winner of the first object), the revenue to the second seller will be very low, indeed, namely k times the second highest valuation. Secondly, the larger the cut-off valuation \hat{v} , the lower is the probability that the first auction is won by the bidder with the lowest valuation. Hence, as \hat{v} increases, it becomes increasingly unlikely that the buy-out price in first auction changes the identity of its winner and, therefore, the price in the second auction.

The second part of (ii) in Lemma 2 can be explained by appeal to the *Revenue Equivalence Theorem*, which states that two mechanisms that result in the same allocation must also give rise to the same overall revenue.¹⁹ Now, the buy-out price changes the identity of the winner of the first auction *only if* both bidders accept the buy-out price *and* the random winner happens to be the low-valuation bidder. Assuming they both accept the buy-out price,

¹⁹See e.g. Klemperer (1999).

we note that if the buy-out price is such that $\hat{v} \in [k\bar{v}, \bar{v}]$, the (random) loser of the first auction must necessarily win the second. To see this, we note that the valuation of the first-auction loser and, hence, his bid in the second auction must be at least \hat{v} . This, in turn, exceeds the rival bid in the second auction which is at most $k\bar{v}$. Thus, when both bidders have valuations above \hat{v} , with $\hat{v} \in [k\bar{v}, \bar{v}]$, each bidder will win precisely one unit. However, the same is true if there is no buy-out price. If both bidders have valuations in the interval $[k\bar{v}, \bar{v}]$, the bidder with the highest valuation wins the first auction, and the other bidder wins the second. In conclusion, when $\hat{v} \in [k\bar{v}, \bar{v}]$ the buy-out price might change the order in which bidders win, but not the final allocation. Consequently, overall revenue is the same with and without a buy-out price.

In contrast, for low values of \hat{v} , $\hat{v} \in [\underline{v}, k\bar{v})$, the presence of a buy-out price might change the final allocation and therefore also overall revenue. In the next subsection we discuss the consequences of this in greater detail.

Proposition 4 (Increasing prices) $ER_2(\hat{v}) > ER_1(\hat{v}), \forall \hat{v} \in [\underline{v}, \bar{v}]$.

Proof. See Appendix A. ■

As remarked in relation to Proposition 1 (Black and de Meza), revenue is strictly increasing over the auction sequence when there is no buy-out price. Indeed, revenue increases with probability one in the case without a buy-out price. However, the result in Proposition 4 is only for *expected* revenues. It is entirely possible that actual, observed revenues decrease when there is a strictly positive buy-out price. For example, if one bidder has a valuation $\hat{v} > 0$ and the other $\underline{v} = 0$, revenue in stage 1 is $B(\hat{v}) > 0$, while revenue in stage 2 is 0. The upshot of Proposition 4 is that the first seller can increase expected revenue by introducing a buy-out price, *but* will *not* be able “to level the playing field”.

3.2 The optimal buy-out price

Now, we move on to determine the *optimal* buy-out price from the perspective of the first seller. Thus the first seller maximizes $ER_1(\hat{v})$, which gives the optimal cut-off. To implement this cut-off the buy-out price is set according to (4) in Proposition 2. Our main result can be stated as follows.

Proposition 5 (i) For $k < 1$ the optimal value of \hat{v} is strictly lower than $k\bar{v}$. Consequently, the sequence of auctions is inefficient when the first seller chooses the buy-out price optimally. (ii) For $k = 1$, $\hat{v} = \bar{v}$ is optimal.

Proof. See Appendix A. ■

This result follows more or less directly from Lemma 2. Since the sum of revenues is the same for all $\hat{v} \in [k\bar{v}, \bar{v}]$, and revenue to the second seller is globally, strictly increasing, it follows that $\hat{v} = k\bar{v}$ dominates all higher cut-off values from the perspective of the first seller. Further, at $\hat{v} = k\bar{v}$ the derivative of $ER_1(\hat{v})$ is strictly negative, and it always pays for the first seller to lower the cut-off valuation below \hat{v} by a suitable choice of the buy-out price B . The consequences for efficiency are immediate: It pays for the first seller to set the buy-out price, B , at such a level that the final allocation is inefficient with strictly positive probability. The optimal first-auction buy-out price is set such that the low-valuation bidder wins the first object with positive probability when he would have won no object in an efficient mechanism.

In the special case where $k = 1$, the behavior in the second auction is independent of the outcome of the first auction. Therefore, stage 1 is essentially equivalent to a one-shot auction. Thus, the last part of Proposition 5 shows that buy-out prices lower revenue in such auctions when buyers are risk neutral.²⁰

At the present level of generality, only the qualitative properties of the solution to the first seller's problem can be established, as captured by Proposition 5. Working out the details, that is, the optimal values of \hat{v} and $B(\hat{v})$, requires a fully specified example.

3.3 Example

Let us reconsider the uniform case with $v \in [0, 1]$. First, we spell out the relationship between the buy-out price, B , and the critical valuation, \hat{v} . In

²⁰For specific distributions, this result has already been noted by Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2003). We show that this is a general property whenever the distribution function is continuously differentiable. Thus, the generality of our argument also reveals that “ironing of marginal revenue” cannot explain the use of buy-out prices in this case (for more on this, see below).

this case (4) can be written as

$$B(\widehat{v})(1 + \widehat{v}) = \begin{cases} k \left(\int_0^1 x dx + \int_0^{\widehat{v}} x dx \right) & \widehat{v} \geq k \\ k \left(\int_0^1 x dx + \int_0^{\widehat{v}} x dx \right) - \int_{\frac{\widehat{v}}{k}}^1 (kx - \widehat{v}) dx & \widehat{v} \leq k \end{cases}$$

which implies that

$$B(\widehat{v}) = \begin{cases} \frac{k}{2(1+\widehat{v})} (1 + \widehat{v}^2) & \widehat{v} \geq k \\ \frac{k}{2(1+\widehat{v})} \left((1 + \widehat{v}^2) - \left(1 - \frac{\widehat{v}}{k}\right)^2 \right) & \widehat{v} \leq k \end{cases}$$

From this we note that $B(\widehat{v}) < \frac{k}{2}$, so that whatever cut-off valuation $\widehat{v} \in [\underline{v}, \bar{v}] = [0, 1]$ we try to implement, the implied buy-out price will always be less than k times the *unconditional* expectation of the value of the first unit won. In the special case referred to above where $k = \frac{1}{3}$, $B(\widehat{v})$ reduces to

$$B(\widehat{v}) = \begin{cases} \frac{1+\widehat{v}^2}{6(1+\widehat{v})} & \widehat{v} \geq \frac{1}{3} \\ \frac{(3-4\widehat{v})\widehat{v}}{3(1+\widehat{v})} & \widehat{v} \leq \frac{1}{3} \end{cases}$$

Hence, if we want to implement a cut-off valuation of $\widehat{v} = \frac{1}{2} > \frac{1}{3} = k$, the buy-out price must be set as $B(\frac{1}{2}) = \frac{5}{36} \approx 0.14$. Similarly, if we want to implement a cut-off valuation of $\widehat{v} = \frac{1}{4} < \frac{1}{3} = k$, the buy-out price must be set as $B(\frac{1}{4}) = \frac{2}{15} \approx 0.13$.

Next, consider the expected revenues given \widehat{v} . In the uniform example, (5) and (6) reduce to

$$ER_1(\widehat{v}) = \begin{cases} \frac{k}{6} (3 - 3\widehat{v} + 3\widehat{v}^2 - \widehat{v}^3) & \widehat{v} \geq k \\ \frac{1}{6k} (6k\widehat{v} - 3(1 + 2k - k^2)\widehat{v}^2 + (3 - k^2)\widehat{v}^3) & \widehat{v} \leq k \end{cases}$$

and

$$ER_2(\widehat{v}) = \begin{cases} \frac{k}{6} (3 - 2k + 3\widehat{v} - 3\widehat{v}^2 + \widehat{v}^3) & \widehat{v} \geq k \\ \frac{1}{6k} ((3 - k)k^2 + 3k(1 - k)\widehat{v}^2 - (1 - k^2)\widehat{v}^3) & \widehat{v} \leq k \end{cases}$$

For the special case where $k = \frac{1}{3}$, the expected revenues can be written as

$$ER_1(\widehat{v}) = \begin{cases} \frac{1}{18} (3 - 3\widehat{v} + 3\widehat{v}^2 - \widehat{v}^3) & \widehat{v} \geq \frac{1}{3} \\ \frac{1}{9} (9\widehat{v} - 21\widehat{v}^2 + 13\widehat{v}^3) & \widehat{v} \leq \frac{1}{3} \end{cases}$$

and

$$ER_2(\widehat{v}) = \begin{cases} \frac{1}{54} (7 + 9\widehat{v} - 9\widehat{v}^2 + 3\widehat{v}^3) & \widehat{v} \geq \frac{1}{3} \\ \frac{1}{54} (8 + 18\widehat{v}^2 - 24\widehat{v}^3) & \widehat{v} \leq \frac{1}{3} \end{cases}$$

Hence, if the buy-out price has been chosen to implement the cut-off valuation $\hat{v} = \frac{1}{2} > \frac{1}{3} = k$, that is, $B \approx 0.14$, the expected revenues are $ER_1(\frac{1}{2}) = \frac{17}{144} \approx 0.12$ and $ER_2(\frac{1}{2}) = \frac{77}{432} \approx 0.18$, and the ratio of expected revenues is $\frac{ER_1(\frac{1}{2})}{ER_2(\frac{1}{2})} = \frac{3672}{5929} \approx 0.62$. Similarly, if the buy-out price has been chosen to implement the cut-off valuation $\hat{v} = \frac{1}{4} < \frac{1}{3} = k$, that is, $B \approx 0.13$, the expected revenues are $ER_1(\frac{1}{4}) = \frac{73}{576} \approx 0.13$ and $ER_2(\frac{1}{4}) = \frac{70}{432} \approx 0.16$, and the ratio of expected revenues is $\frac{ER_1(\frac{1}{4})}{ER_2(\frac{1}{4})} = \frac{1971}{2695} \approx 0.73$. When pitted against the first auction revenues in two straight second-price auctions, it is, thus, clear how the first seller can raise his revenue by introducing a buy-out price.²¹

Finally, turn to the optimal value of the cut-off. From Proposition 5 we know that $\hat{v} < k\bar{v} = k$, for any $k \in (0, 1)$. The expected revenue to the first seller when $\hat{v} < k$ is given by

$$ER_1(\hat{v}) = \frac{1}{6k}((3 - k^2)\hat{v}^3 - 3(1 + 2k - k^2)\hat{v}^2 + 6k\hat{v})$$

while the expected revenue to the second seller is

$$ER_2(\hat{v}) = \frac{1}{6k}(-(1 - k^2)\hat{v}^3 + 3k(1 - k)\hat{v}^2 + (3 - k)k^2)$$

Maximizing $ER_1(\hat{v})$ with respect to \hat{v} gives the optimal cut-off valuation from the perspective of the first seller

$$v^* = \frac{1 + 2k - k^2}{3 - k^2} - \frac{((1 + 2k - k^2)^2 - 2k(3 - k^2))^{1/2}}{3 - k^2} < k = k\bar{v}$$

and the associated, optimal buy-out price, $B(v^*)$ is given by

$$B(v^*) = \frac{k}{2(1 + v^*)} \left((1 + (v^*)^2) - \left(1 - \frac{v^*}{k}\right)^2 \right)$$

We can substitute v^* into the revenue expressions, and Fig. 3 illustrates how $ER_1(v^*)$ (*thin*) and $ER_2(v^*)$ (*heavy*) vary with k .

²¹Recall from the previous section that $ER_1^{SSP} = \frac{1}{9} \approx 0.11$ and $ER_2^{SSP} = \frac{5}{27} \approx 0.19$ when $k = \frac{1}{3}$.

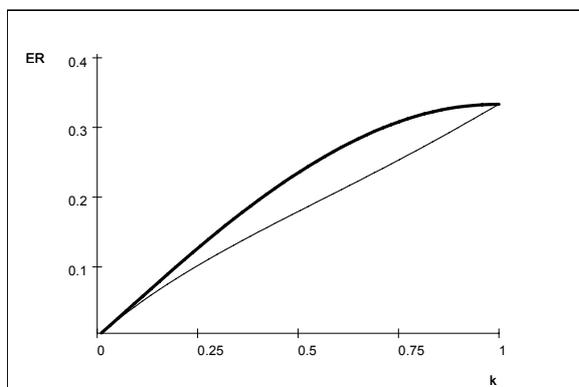


Fig. 3: Revenues in auction with buy-out

The ratio between the expected revenues given an optimally chosen buy-out price, $RR(BO) = \frac{ER_1(v^*)}{ER_2(v^*)}$, is illustrated in the following figure

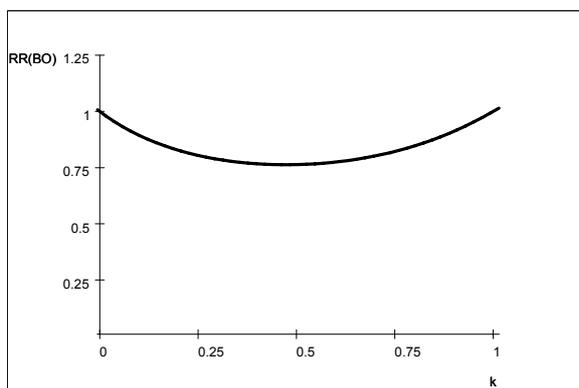


Fig. 4: Revenue Ratio in Auction with Buy-out

We can compare with the case of two straight second-price auctions illustrated in Fig. 1 and Fig. 2. In Fig. 5 we merge the information in Fig. 1 and Fig. 3. The dashed lines are for two straight second-price auctions, while the solid lines are for the case where the first seller chooses the buy-out price to implement v^* .

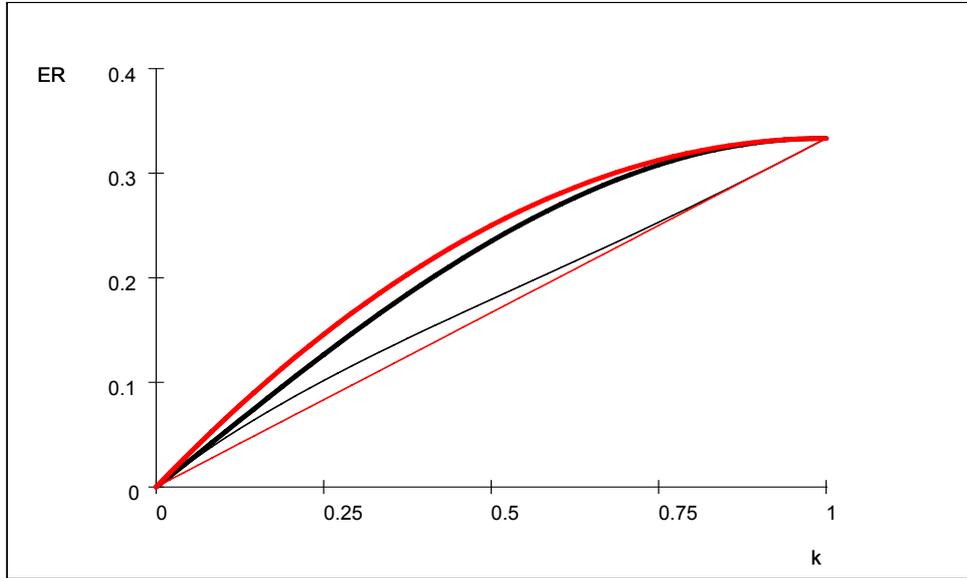


Fig. 5: Comparison of auction revenues

Fig. 6 merges the information from Fig. 2 and Fig. 4, and the *thin* line is for two straight second-price auctions, while the *heavy* line is associated with an optimal buy-out price.

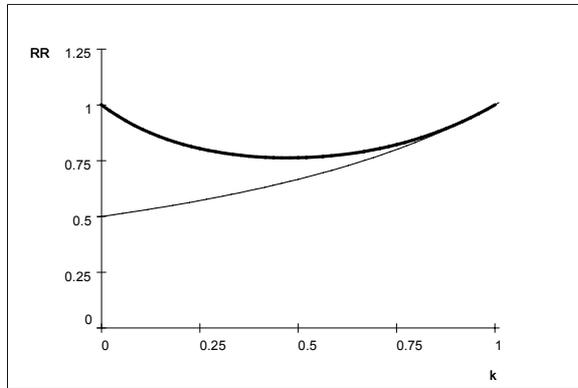


Fig. 6: Revenue Ratios

Finally, in Fig. 7 we plot the percentage gain to the first seller from an optimally chosen buy-out compared to the straight second-price auction, $G = 100 \times \frac{ER_1(v^*) - ER_1^{SSP}}{ER_1^{SSP}}$.

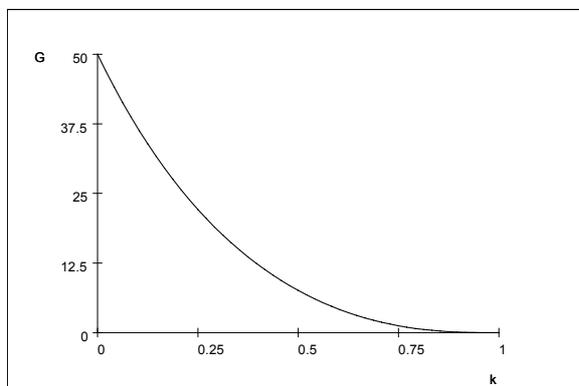


Fig. 7: Percentage gain from buy-out price

The last three figures essentially illustrate that the value from the perspective of the first seller of introducing a buy-out price is substantial when the individual demand functions are relatively steep (k small). When demands are steep, and there are only two bidders, the competition for the first object will be weak. It follows that the first seller has a strong incentive to try to improve his position in this case by introducing a suitably chosen buy-out price. The following table captures central features of the example in an alternative way.

k	ER_1^{SSP}	v^*	$B(v^*)$	$ER_1(v^*)$	G
0.01	0.00333	0.00995	0.00495	0.00495	48.65
0.10	0.03333	0.09549	0.04597	0.04558	36.75
0.25	0.08333	0.22618	0.10623	0.10176	22.12
0.50	0.16667	0.43308	0.20404	0.17931	7.58
0.75	0.25000	0.66667	0.32222	0.25309	1.24

Recall that in this example revenue equivalence and efficiency is lost when \hat{v} is set below $k = k\bar{v}$. Hence, a comparison of the first and third column is indicative of the inefficiency when \hat{v} is set optimally. For example, when $k = k\bar{v} = \frac{1}{2}$ the optimal \hat{v} is approximately 0.43, which implies that there is a small, but “non-trivial”, probability that the final allocation is inefficient. Note that $k = \frac{1}{2}$ implies that $ER_2^{SSP} - ER_1^{SSP} = \frac{1}{3}k(1 - k)$ is maximized. When the first seller sets the optimal buy-out price $B(v^*) \approx 0.2$, he manages to increase his expected revenue by 7.58%, while aggregate revenues fall by only 0.58%.

4 Buy-Outs, Inefficiency and Revenue Non-Equivalence

In the next section, we assume that the two objects are owned by a *single* seller and show that a buy-out price in the *last* auction is beneficial to this seller. Before proceeding, however, it is of some value to examine more closely why overall revenue declines when a buy-out price is offered by the *first* of *two* sellers.

As mentioned, the *Revenue Equivalence Theorem* reveals that if two mechanisms yield the same allocation, expected revenue in the two mechanisms must also be the same. Since the outcome of the bench-mark model is efficient, it follows that introducing a buy-out price changes total revenue if and only if²² the resulting allocation is inefficient.

For instance, introducing a buy-out price in the *first* auction results in the following kind of inefficiency: an agent may win one item when he would have won none without the buy-out price. In the next section, a buy-out price in the *second* auction will be shown to cause another type of inefficiency: an agent may win both items, when he would have won exactly one without a buy-out price. In the latter case, an agent who would have won one unit in an efficient mechanism risks not winning one at all. In this sense the type of inefficiency studied in the next section is the opposite of that studied above.

To understand the consequences of these different kinds of inefficiencies, it is useful to exploit the similarities between monopoly pricing and auctions²³. When a monopolist faces agents with multi-unit demands, it is well known that the optimal pricing schedule may involve quantity discounts. These discounts enable the monopolist to sell several units to agents with high *marginal revenue* on all units, without at the same time selling to agents with low marginal revenue on some units. Whether agents have unit or multi-unit demands, it is well understood that the key ingredient in the monopolist's optimization problem is marginal revenue.

Now, the expression for what amounts to marginal revenue of a bidder

²²This assumes that an agent of type \underline{v} is indifferent between the two mechanisms. We will return to this point momentarily.

²³These similarities were first pointed out by Bulow and Roberts (1989) for auctions with unit demand, see also Bulow and Klemperer (1996) and Klemperer (1999). Maskin and Riley (1989) draw parallels between auctions with multi-unit demand and non-linear pricing. For more on the latter, see also Kirkegaard (2003).

with valuation v in an auction is

$$J(v) = v - \frac{1 - F(v)}{f(v)}$$

for the first unit, and it can easily be shown that marginal revenue is $kJ(v)$ for the second unit.²⁴ The expected revenue to the seller is then

$$E \left[\sum_{i=1}^2 (q_i^1(v_1, v_2)J(v_i) + q_i^2(v_1, v_2)kJ(v_i)) \right] - 2EU(\underline{v}, \underline{v}) \quad (7)$$

where $q_i^j(v_1, v_2)$ is the probability that agent i wins at least j units, given that the two agents are of type v_1 and v_2 , respectively. The last term is the expected rent obtained by an agent of type \underline{v} in the mechanism. (7) is the counterpart of the revenue for a monopolist, who earns the area under the marginal revenue curve.

Clearly, if $EU(\underline{v}, \underline{v})$ is the same across different mechanisms, and if these mechanisms implement the same allocation, (i.e., the same $q_i^j(v_1, v_2)$), expected revenue must be the same. This is the *Revenue Equivalence Theorem*.

We are now equipped to provide an alternative proof of why overall revenue declines when an optimally chosen buy-out price is introduced by the first seller. Given that $\underline{v} < \hat{v} < k\bar{v}$, the allocation changes as a consequence of the buy-out price, if the winner of stage 1 would not have won a unit at all in the efficient allocation. If the winner of stage 1 has valuation v , this happens when $v < k\bar{v}$, and if the rival bidder has valuation $x \in (\frac{v}{k}, \bar{v})$. In this case the revenue *gain* from the winner of the first auction (who should have won no item) is simply $J(v)$. The revenue *loss* from the loser of the first auction (who will only win the second item, when he should have won both) is $E(kJ(x) \mid x > \frac{v}{k})$, which we can write as

$$\int_{\frac{v}{k}}^{\bar{v}} kJ(x) \frac{f(x)}{1 - F(\frac{v}{k})} dx = v$$

Now, $J(v) < v$, and we conclude that, given the event that the allocation has changed, the marginal revenue gained falls short of the marginal revenue

²⁴For a derivation of $J(v)$, see Myerson (1981) or Bulow and Roberts (1989). Since willingness-to-pay for a second unit is k times that for the first unit, it is unsurprising that marginal revenue of the second unit is k times marginal revenue of the first unit, see Kirkegaard (2003).

lost. Thus, overall revenue decreases since the first term in (7) declines, while second term is unchanged. Hence, it is not profitable to allow an agent to win *one* unit too often, compared to the efficient allocation.

4.1 Unit demands

The analogy between auction and monopoly, and (7) in particular, allows an easy explanation of why multi-unit demands ($k > 0$) are necessary to motivate the use of a buy-out price in stage 1. Assume to the contrary that $k = 0$, or that bidders have unit demands. To make the problem interesting, assume that there are $n > 2$ buyers, implying that there is excess demand. For simplicity, assume also that $J(\cdot)$ is strictly increasing.

Without loss of generality, label the bidders in descending order of valuations, $v_1 \geq v_2 \geq \dots \geq v_n$. Then, in two straight second-price auctions, bidder 1 wins the first auction, and bidder 2 the second auction. Hence, the sequence of auctions is efficient. In the second auction, the price is v_3 .

With a buy-out price, bidder 1 may lose stage 1 to another bidder with valuation above the cut-off, \hat{v} . However, if bidder 1 loses stage 1, he is sure to win stage 2. Nevertheless, the sequence of auctions is not necessarily efficient, because bidder 2 is not guaranteed to win any item. By inspecting (7) and recalling that $J(\cdot)$ is monotonic, it is clear that overall revenue must be lower with the buy-out price than without.

Next, let us examine revenue in stage 2. If bidder 1 or bidder 2 won the first stage, the price in stage 2 will be v_3 . However, if neither bidder 1 nor bidder 2 won the first stage, the price in stage 2 will clearly be v_2 , $v_2 \geq v_3$. Hence, the second seller is better off with a buy-out price in stage 1.²⁵ Since overall revenue also decreases, we conclude that the first seller is worse off by offering a buy-out price.²⁶

²⁵To minimize the probability that either bidder 1 or bidder 2 wins stage 1, the second seller would ideally want the first seller to set a buy-out price of zero (“give away the first item for free”).

²⁶Note that these remarks apply whenever Assumption 1 is violated, that is, $k\bar{v} \leq \underline{v}$. Hence, for a stage 1 buy-out price to make sense, even with $n > 2$ bidders, there must be “effective” multi-unit demand ($k\bar{v} > \underline{v}$).

4.2 One-shot auctions

We have already argued that when $k = 1$ (horizontal demands), stage 1 is equivalent to a one-shot auction. In one-shot auctions, revenue is clearly maximized by allocating the object to the agent with the highest marginal revenue. When the agent with the highest valuation is also the agent with the highest $J(v)$, that is, when $J(v)$ is increasing in v , this is accomplished with an efficient mechanism. However, when $J(v)$ is non-monotonic, it is impossible to always give the object to the agent with the highest marginal revenue. The reason is that the auctioneer must respect the incentive compatibility constraints when designing his mechanism. To satisfy these, it is necessary that the probability of winning the object is non-decreasing in the valuation.

In the cases where $J(v)$ is non-monotonic, the rules of the optimal mechanism²⁷ ensure that the probability of winning is constant over a subset of valuations. That is, agents with different valuations have the same probability of winning, and therefore contributes marginally the same to revenue. Hence, the optimal mechanism is said to “iron” the marginal revenue curve. Now, we observe that the buy-out price is a crude way of ironing the marginal revenue curve, since all agents with valuation above \hat{v} have the same probability of winning in a one-shot auction. It is crude because the interval on which marginal revenue is ironed in an optimal mechanism is always interior, whereas the buy-out price also bundles valuations close to and including \bar{v} with lower valuations.

Since buy-out prices offer some (excessive) ironing, it is perhaps not obvious whether or not buy-out prices can increase revenue when $J(v)$ is non-monotonic and $k = 1$. However, our model is sufficiently general to encompass these situations, and we can therefore conclude that buy-out prices are counterproductive even when some ironing is called for, precisely because the ironing is too crude. We stress this, since we are not aware of any papers on auctions (or monopoly) showing that “ironing” may be counterproductive, if it is too crude in the sense of this paper. Among the related papers the model of Budish and Takeyama (2001) is discrete, while Reynolds and Wooders (2003) assume uniformly distributed valuations. Ironing is not an issue in either of these specifications. Mathews (2002) also assumes uniform distributions, but he remarks that his results hold for any distribution, though without referring to ironing.

²⁷See Myerson (1981) or Bulow and Roberts (1989).

Thus, in one-shot auctions, buy-out prices are unprofitable when utilized in the way assumed so far, even if the optimal auction involves ironing of the marginal revenue curve. However, a more sophisticated design can combine buy-out prices and reserve prices to maximize revenue in these cases.

As mentioned, the optimal interval on which ironing should be performed is interior. Let this interval²⁸ be given by $[\hat{v}, v^r]$, $\underline{v} < \hat{v} < v^r < \bar{v}$. Consider the following auction for one object, which takes place in two stages. First, an auction with a reserve price is staged. If the object is not sold, another auction is staged, in which a buy-out price is available. The reserve price and the buy-out price should jointly be set in such a way that a bidder bids in the first stage if and only if his valuation is larger than v^r , and such that the buy-out price in stage 2 is accepted if and only if the bidder has a valuation that exceeds \hat{v} .^{29,30}

Now, the auction is efficient if at most one bidder has a valuation in the interval $[\hat{v}, v^r]$. Otherwise, however, both bidders will not participate in the first stage, but will instead accept the buy-out price in stage 2. Hence, the bidders have an even chance of winning, and this chance is, importantly, independent of the exact valuation. In other words, the marginal revenue curve has been optimally ironed, and it follows that the proposed two stage auction maximizes revenue.³¹

4.3 Permanent buy-out prices

In this paper we have chosen to focus on *eBay*'s version of the buy-out price. The buy-out price is temporary on *eBay*, whereas *Yahoo!* offers a permanent buy-out price (termed the *Buy Price*). Reynolds and Wooders (2003) compare the two types of buy-out prices³² in one-shot auctions.

²⁸We assume there is only one such interval. However, it should be obvious how to extend the following mechanism if there are more.

²⁹To achieve this, it is necessary that the buy-out price in stage 2 is known before stage 1 commences. This form of precommitment by the seller is discussed further in Section 5.

³⁰It is straightforward to show that such a combination of a reserve price and a buy-out price exists.

³¹To be precise, the auction is optimal among all auctions that sell the object with probability one. Notice that the reserve price in this auction does not serve to ration output (see footnote 10). Obviously, such a reserve could be added to the second stage of the auction.

³²Reynolds and Wooders (2003) assume there are two bidders. Hidvégi, Wang and Whinston (2003) analyze the permanent buy-out price with an arbitrary number of bid-

With a permanent buy-out price, bidders with very high valuations may accept the buy-out price immediately. Bidders with lower valuations initially ignore the buy-out price, but as bidding in the English auction progresses, they become more and more pessimistic about the severity of the competition and eventually accept the buy-out price (given that it is lower than the valuation). The higher the valuation, the sooner the buy-out price is accepted.

If the buy-out price is very high, not even a bidder with valuation \bar{v} will accept it immediately. As the price in the English auction increases, however, the buy-out price may be accepted. If so, it is accepted by the bidder with the highest valuation, and the auction is efficient. On the other hand, if the buy-out price is such that it would be accepted by a bidder with valuation in the interval $[\hat{v}, \bar{v}]$ in an *eBay* auction, the same bidder accepts it immediately in the *Yahoo!* auction. If it is not accepted immediately, it may be accepted later on, by the high valuation bidder. Hence, $B(\hat{v})$ causes the same type of inefficiency whether it is a permanent or temporary buy-out price.

By the *Revenue Equivalence Theorem*, the *eBay* and *Yahoo!* auction formats yield the same revenue, for a given $B(\hat{v})$. Thus, our results are also valid when the buy-out price is permanent. In fact, we conjecture that the results thus far are robust to small changes in the extensive form of the game. The reason is straightforward, and relies only on the possibility that the buy-out price may cause inefficiencies.

In particular, assume that in the equilibrium (on which bidders coordinate) of the particular game, there is a \hat{v} such that stage 1 is won by the bidder with the highest valuation, if at most one bidder has a valuation that exceeds \hat{v} , but that there is a strictly positive probability that it is won by the other bidder otherwise. In this event, the second seller is worse off (because competition is diminished in stage 2). At the same time, however, we know that overall revenue is unchanged if $\hat{v} \geq k\bar{v}$ (because the sequence of auctions is efficient in this case). Consequently, when \hat{v} is sufficiently high, the first seller is better off with a buy-out price as long as the identity of the winner in stage 1 changes with positive probability.

To conclude this section, we note, quite generally, that overall revenue is adversely affected by the buy-out price, if the inefficiency is of the form that an agent wins one unit more often than is efficient. In the next section, however, we show that it is possible to increase revenue by introducing another

ders.

form of inefficiency.

5 One Seller

In the following, we assume that the same seller owns both objects, and that they are sold sequentially. Above we established that total revenue decreases if a buy-out price is offered in the first auction, because an undesirable kind of inefficiency was generated. However, in the following we show that a buy-out price in the second auction produces a different type of inefficiency, one which is desirable for the seller. To this end, we consider the following augmented game:

- 1 The first object is sold using a second-price auction. The closing price is observed.
- 2 The seller announces a buy-out price, B , for the second object. The object is sold at the price B if at least one bidder bids B . If both bidders bid B , one bidder is picked at random to win. If no one bids B , a normal second-price auction is staged. The price can exceed B in this event.

In line with much of the literature on mechanism design, we will accord the seller a powerful ability to pre-commit to a particular auction design. To illustrate, suppose the first auction is conducted, and the closing price is observed. Hence, if bidding strategies in the first auction are *strictly increasing*, the valuation of the loser, v , say, is revealed. Contingent on this v , a buy-out price for the second auction, $B(v)$, is set. We assume throughout, and this is where commitment matters, that the relation between v and B is firmly understood by bidders at the outset. Thus, the seller can credibly announce $B(v)$ before the first auction.³³

Given this set-up, our basic argument can be outlined as follows. Assume that the bidding strategy in the first auction is strictly increasing, and that the closing price, p , is observed. Since the latter is determined by the bidding strategy of the runner-up, the valuation of this agent, v , can be deduced. Then, in the second stage, a buy-out price is offered, which is contingent on v . Assuming that the buy-out price, B , is close to v , it is not desirable for

³³For more on this, see below.

the *loser* of stage 1 to accept it.³⁴ However, if B is lower than v , the *winner* of stage 1 will accept it, if his willingness-to-pay exceeds the buy-out price. The reason is that if he does not, a second-price auction ensues, in which he knows the loser of stage 1 will be willing to compete for the object until the price reaches $v > B$. Consequently, the winner of stage 1 also wins stage 2 if his valuation is at least B , although he would win less often in an efficient auction, namely when his valuation is above v .

To see why this might increase revenue, observe that it is common for a monopolist to offer quantity discounts. These discounts introduce the same kind of inefficiency as that described above. If p is the price of one unit and $p + B < 2p$ the price of two units, an agent may be willing to pay more than B for one unit, but less than p . That is, $B < v < p$. In this case, he will obviously not buy a single unit. If, in addition, $p + B$ exceeds the value of two units, this agent will not buy two units either. On the other hand, a buyer willing to pay exactly p for one unit and an additional B for a second unit will purchase two units. Clearly, it would be efficient for these two buyers to share the two units. By introducing the inefficiency, however, the monopolist is able to sell to the agent with highest marginal revenue on the incremental unit.

To close the argument, we need to understand why this kind of inefficiency favors agents with high marginal revenue. The first observation is that $kJ(\bar{v}) > J(k\bar{v})$, implying that the agent with the highest possible valuation should win two units, even when faced with a competitor with valuation slightly higher than $k\bar{v}$, and even though this is inefficient.

Hence, inefficiency “at the top” is always desirable from the point of view of revenue generation. Often, however, inefficiency is also desirable at all other levels. Assume for the rest of the section that the following monotonicity condition is satisfied.

Assumption 2. $\frac{1-F(v)}{f(v)}$ is decreasing in v .

This increasing *hazard rate*³⁵ condition implies (but is not implied by) an increasing $J(\cdot)$ function (i.e., decreasing marginal revenues in the more

³⁴This is because the potential gain, $v - B(v)$, is small, and the stage 1 loser prefers to participate in a straight second-price auction in stage 2.

³⁵The hazard rate is $h(v) = \frac{f(v)}{1-F(v)}$. An increasing hazard rate is equivalent to log-concavity of $1 - F(v)$. See Bagnoli and Bergstrom (1989) for an extensive treatment of log-concave distributions.

familiar context). A consequence of this is that, for $kv \geq \underline{v}$,

$$k \frac{1 - F(v)}{f(v)} < \frac{1 - F(kv)}{f(kv)} \iff kJ(v) > J(kv)$$

such that a bidder with valuation v should win two units when faced by a rival with a valuation in a neighborhood of kv .

Thus, the seller would like to design an auction such that a bidder with valuation $v \in [\frac{\underline{v}}{k}, \bar{v}]$ wins two units when faced by a rival with valuation close to kv , that is, he wins two units more often than is efficient. As argued in the beginning of this section, this can be accomplished by using a buy-out price in the second auction.

To elaborate, if v is the revealed valuation of the stage 1 loser, we consider a commonly known function $B(v)$ which gives the resulting buy-out price in stage 2. That is, $B(v)$ is known before the first auction commences. The buy-out price is assumed to satisfy $B(v) \leq v$ for all v . We will then look for a discriminating equilibrium, defined as follows.

Definition 1 *A discriminating equilibrium consists of a symmetric bidding strategy in stage 1, which is strictly increasing in bidder valuation, and the following strategy in stage 2. Given that $B(v)$ is the buy-out price in stage 2, the winner of stage 1 bids $B(v)$ in stage 2 if and only if his valuation of the second unit exceeds $B(v)$, while the loser of stage 1 never bids $B(v)$.³⁶ Bidder i , $i = 1, 2$, bids his (marginal) valuation in stage 2, if the buy-out price was not accepted by anyone.*

Inspection of Definition 1 reveals that the existence of a discriminating equilibrium necessitates that $\underline{v} > 0$. To see this, assume to the contrary that $\underline{v} = 0$ and consider the incentives of a bidder with a valuation slightly above 0. By following the equilibrium strategy, it is very unlikely that such a bidder will win either auction. Rather, it is preferable for such an agent to bid 0 in the first auction, and then accept the buy-out price (of zero) in the second auction. Since the competing agent will also want to buy the good in the second auction at the buy-out price, the low-valuation agent wins the second auction with a significant probability of 0.5.

On the other hand, if $\underline{v} > 0$ and $B(v)$ is close to v , an agent with a valuation close to \underline{v} prefers not to accept B if he lost stage 1, even if he

³⁶Note that it is only along the equilibrium path that the loser of stage 1 never accepts $B(v)$ in stage 2. We do not look for a specific off-the-equilibrium path behavior.

deviated in the first auction. The reason is that there is a mass of types for which $kv < B$, implying that the low valuation agent wins the second auction with significant probability and pays significantly less than his own valuation when following the equilibrium strategy. This is preferable to accepting the buy-out price and winning with an even larger probability, provided that the buy-out price is large relative to the valuation.

These arguments capture the key qualitative difference between cases with $\underline{v} = 0$ and $\underline{v} > 0$. When $\underline{v} = 0$, a bidder with valuation \underline{v} does not contribute to the competition for any of the units since $\underline{v} < kv, \forall v > \underline{v}$.³⁷ In contrast, when $\underline{v} > 0$, even a bidder with the lowest possible valuation, \underline{v} , contributes to the competition, since there is a range of v such that $kv < \underline{v}$.³⁸

If the first auction was won by bidder 1, say, the winner of the second stage changes as a consequence of the buy-out price if and only if $B(v_2) < kv_1 < v_2$. In this case, bidder 1 also wins the second item, resulting in the desired inefficiency. The seller seeks to construct a $B(v)$ function which has the following properties.

Assumption 3. Define $B(v)$ on $[\underline{v}, \bar{v}]$ and make the following assumptions.

- (i) $B(v) \in (kv, v), \forall v \in (\underline{v}, k\bar{v}), B(v) = v$ otherwise³⁹. $B(v)$ is everywhere continuous, and it is continuously differentiable with $0 < B'(v) < \infty, \forall v \in [\underline{v}, \bar{v}] \setminus \{k\bar{v}\}$.
- (ii) $kJ(x) > J(v), \forall x \geq \frac{1}{k}B(v)$.
- (iii) The function $b(v)$ is strictly increasing⁴⁰, where

$$b(v) = \begin{cases} kv + (v - B(v)) \frac{f(\frac{1}{k}B(v))}{f(v)} \frac{1}{k} B'(v) & \text{for } v \in [\underline{v}, k\bar{v}) \\ kv & \text{for } v \in [k\bar{v}, \bar{v}]. \end{cases}$$

We will show below that the function $b(v)$ is the bidding strategy in stage 1 of a discriminating equilibrium. If the loser of stage 1 is revealed to be of

³⁷A bidder with valuation \underline{v} could never be expected to win even in the competition for the second item. This is easily checked against the results of Section 2.

³⁸In this case, a bidder with valuation \underline{v} could reasonably win the second unit. Again, this can easily be checked against Section 2.

³⁹If the agent who loses stage 1 deviated to an action that is not played by any type in equilibrium, this is taken to signal that $v = \underline{v}$.

⁴⁰ $b(v)$ is continuous given part (i).

type $v \in [\underline{v}, k\bar{v})$, the buy-out price in stage 2 is $B(v) < k\bar{v}$, and it is accepted with strictly positive probability. However, if the loser is of a higher type, $B(v)$ exceeds $k\bar{v}$, and there is therefore zero probability that the winner of stage 1 accepts it. Note that the ability to precommit to the auction design is formally important, as the design is not time consistent. Once stage 2 is reached, it is no longer in the seller's interest to offer the buy-out price, since this will decrease revenue in stage 2.

Before stating the result of this section, we observe that the second term of (7) is unchanged. This is because a bidder with valuation \underline{v} will lose stage 1 in both auction formats, and since $B(\underline{v}) = \underline{v}$ the presence of the buy-out price in stage 2 will not affect the probability of such a bidder winning (which will be $F(\frac{\underline{v}}{k})$) or the price paid in that event.⁴¹ Furthermore, while we argued that inefficiency at the top is always desirable, we explicitly assumed that $B(k\bar{v}) = k\bar{v}$, implying that there is no inefficiency at the top. This part of Assumption 3 is made solely to simplify the proof of the following proposition.

Proposition 6 (i) *Any discriminating equilibrium satisfying Assumption 3 is strictly revenue superior to the equilibrium of a sequence of second-price auctions with no buy-out price.* (ii) *A discriminating equilibrium satisfying Assumption 3 exists whenever $\underline{v} \geq kE(v)$. In such an equilibrium, the bidding strategy in stage 1 is given by $b(v)$.*

Proof. (i) The proof is based on inspection of (7). As mentioned above, the second term is unchanged. However, the first term in (7) is higher when the buy-out price is introduced. To see this, observe that if the allocation changes, the winner of stage 1 must have a type that exceeds $\frac{\underline{v}}{k}$. By the second part of Assumption 3, the marginal revenue of the second unit to this bidder is higher than the marginal revenue of the first unit to the losing bidder. Hence, for every realization of (v_1, v_2) , the term inside the expectations operator in (7) is no lower, but possibly higher, than without the buy price. For a proof of the second part of the proposition, see Appendix A. ■

The condition in the second part of Proposition 6 is required to eliminate any incentive to bid *low* in stage 1, and then (contrary to the supposition)

⁴¹In fact, this is why this assumption has been imposed. We could let $B(\underline{v}) < \underline{v}$, which would decrease the second term in (7) and hence increase revenue further. However, we seek the stronger result that it is the change in allocation (i.e. the inefficiency) that drives revenue up. Thus, we keep the second term the same over the two auction formats.

to accept the buy-out price in stage 2 if stage 1 was lost. In fact, ruling out *sizeable, downward* deviations from $b(v)$ is the difficult part of the proof of existence, while local deviations and any upward deviation are easily ruled out. It follows immediately from the second part of the proposition that the presence of a buy-out price in stage 2 intensifies bidding competition in stage 1, since $b(v) \geq kv$. Thus, revenue in stage 1 increases. The sum of revenues in the two stages also increases, despite the fact that revenue in stage 2 decreases.

We also observe that Assumption 1 implies that k cannot be too small, while condition (ii) in Proposition 6 implies that k cannot be too great either, that is, $k \in \left(\frac{v}{\bar{v}}, \frac{v}{E(v)}\right]$.⁴² As an example, the assumptions are satisfied for the uniform distribution on $[1, 2]$ with $k \in \left(\frac{1}{2}, \frac{2}{3}\right)$.

Finally, we note that the conclusion that a discriminating auction (second-stage buy-out) may increase overall revenue of the seller is related to a further result in Black and de Meza (1992). They show, by an example, that an option *offered to the first-round winner* of buying the second object at the first-round price may increase overall revenue above the level of two straight second-price auctions. Despite the one-sided nature of the option suggested by Black and de Meza it, presumably, trades on the same type of inefficiency as in this section. That is, the winner of the first round wins more often than is efficient.

6 Concluding Remarks

In this paper we sought to explain the use of buy-out prices by observing that online auction markets are dynamic, with players knowing that goods not presently on the market are likely to be offered in the future. It was shown that there is an incentive for current sellers to offer a buy-out price that is accepted with positive probability. Furthermore, we showed that a sophisticated seller with several units can increase the sum of revenues by introducing a buy-out price in later auctions which is contingent on the outcome of earlier auctions.

⁴²Note that these are sufficient conditions.

References

- Bagnoli, M., and T. Bergstrom, 1989, Log-Concave Probability and Its Applications, draft, University of Michigan.
- Black, J., and D. de Meza, 1992, Systematic Price Differences Between Successive Auctions Are No Anomaly, *Journal of Economics and Management Strategy* 1: 607-628.
- Budish, E., and L. Takeyama, 2001, Buy Prices in Online Auctions: Irrationality on the Internet?, *Economics Letters* 73: 325-333.
- Bulow, J., and P. Klemperer, 1996, Auctions vs. Negotiations, *American Economic Review* 86: 180-194.
- Bulow, J., and J. Roberts, 1989, The Simple Economics of Optimal Auctions, *Journal of Political Economy* 97: 1060-1090.
- Hidvégi, Z., W. Wang and A. Whinston, 2003, Buy-Price English Auction, draft, University of Texas, Austin.
- Katzman, B., 1999, A Two Stage Sequential Auction with Multi-Unit Demands, *Journal of Economic Theory* 86: 77-99.
- Kirkegaard, R., 2003, Inefficiency and Non-Linear Pricing in the Optimal Multi-Unit Auction, draft, University of Aarhus.
- Klemperer, P., 1999, Auction Theory: A Guide to the Literature, *Journal of Economic Surveys* 13: 227-286.
- Lucking-Reiley, D., 2000, Auctions on the Internet: What's Being Auctioned and How?, *Journal of Industrial Economics* 48: 227-252.
- Maskin, E., and J. Riley, 1989, Optimal Multi-Unit Auctions, in F. Hahn (ed.), *The Economics of Missing Markets, Information and Games*, Oxford University Press, UK: Oxford.
- Mathews, T., 2002, Buyout Options in Internet Auction Markets, unpublished Ph.D. thesis, SUNY, Stony Brook.
- Myerson, R., 1981, Optimal Auction Design, *Mathematics of Operations Research* 6: 58-73.
- Reynolds, S., and J. Wooders, 2003, Ascending Bid Auctions with a Buy-Now Price, WP #2003-01, University of Arizona, Tucson.

Weber, R., 1983, Multiple-Object Auctions, in R. Engelbrecht-Wiggans, M. Shubik and R. Stark (eds.), *Auctions, Bidding and Contracting: Uses and Theory*, New York University Press, NY: New York.

Appendix A

Proof of Proposition 2. Consider a bidder with valuation $v \geq \hat{v}$. By bidding B in stage 1, his expected payoff in the two stages is

$$\begin{aligned} EU(B, v) &= \int_{\underline{v}}^{\hat{v}} (v - B)f(x)dx + \int_{\underline{v}}^{\min\{\hat{v}, \max\{\underline{v}, kv\}\}} (kv - x)f(x)dx \\ &+ \int_{\hat{v}}^{\bar{v}} \frac{1}{2}(v - B)f(x)dx + \int_{\hat{v}}^{\min\{\bar{v}, \frac{v}{k}\}} \frac{1}{2}(v - kx)f(x)dx \\ &+ \int_{\hat{v}}^{\max\{\hat{v}, kv\}} \frac{1}{2}(kv - x)f(x)dx \end{aligned}$$

where the five terms capture all the possible outcomes as follows. *First*, the bidder wins stage 1 at a price of B with probability one, if the competitor refrains from accepting B , i.e. has valuation below \hat{v} . *Second*, with probability one, the bidder wins stage 2 at a price equal to the valuation of his rival, if this rival did not accept B in stage 1 (she has a valuation below \hat{v}), and if her bid, or valuation, (which exceeds \underline{v}) is at most kv . *Third*, the first auction is won with probability 0.5 if the opponent also bids B , i.e. if she has a valuation above \hat{v} . *Fourth*, if the player lost stage 1 because the other player also bid B , the second stage is won at a price equal to the rival's bid if this bid is not too high. *Finally*, if both players bid B in stage 1 and the player in question won, we deduce that the competitor's valuation is at least \hat{v} , implying that the second auction is also won if the rival's valuation is nevertheless so low that the winner of stage 1 will submit a higher bid than the loser.

If the bidder, instead, does not bid B , the first unit will be sold at a second-price auction, if the buy-out price is not accepted by the rival either. The best response in this subgame is easily shown to be to outbid the other bidder (the bidder in question is willing to bid kv , whereas the other bidder is known to be willing to bid at most $k\hat{v}$, if she did not bid B right away). Hence, by not bidding B , expected payoff is

$$\begin{aligned} EU(NB, v) &= \int_{\underline{v}}^{\hat{v}} (v - kx)f(x)dx + \int_{\underline{v}}^{\min\{\hat{v}, \max\{\underline{v}, kv\}\}} (kv - x)f(x)dx \\ &+ \int_{\hat{v}}^{\min\{\bar{v}, \frac{v}{k}\}} (v - kx)f(x)dx \end{aligned}$$

Letting $B(\hat{v})$ be the buy-out price at which type \hat{v} is indifferent between these two strategies yields (4). In general, for $v \geq \hat{v}$,

$$\begin{aligned}
& EU(B, v) - EU(NB, v) \\
&= \int_{\underline{v}}^{\hat{v}} (kx - B)f(x)dx + \int_{\hat{v}}^{\bar{v}} \frac{1}{2}(v - B)f(x)dx \\
&\quad - \int_{\hat{v}}^{\min\{\bar{v}, \frac{v}{k}\}} \frac{1}{2}(v - kx)f(x)dx + \int_{\hat{v}}^{\max\{\hat{v}, kv\}} \frac{1}{2}(kv - x)f(x)dx
\end{aligned} \tag{8}$$

the derivative of which is

$$\frac{1}{2} \left[1 - F(\hat{v}) - \left(F(\min\{\bar{v}, \frac{v}{k}\}) - F(\hat{v}) \right) + k \left(F(\max\{\hat{v}, kv\}) - F(\hat{v}) \right) \right] \geq 0$$

Since $EU(B, \hat{v}) - EU(NB, \hat{v}) = 0$ by construction, it follows that $EU(B, v) - EU(NB, v) \geq 0$ for all $v \geq \hat{v}$. Hence, players with high valuations have no incentive to deviate from the equilibrium strategy.

For agents of type $v < \hat{v}$, the equilibrium strategy of not bidding B followed by bidding kv in stage 1 if the opponent did not bid B either, yields the following

$$\begin{aligned}
EU(NB, v) &= \int_{\underline{v}}^v (v - kx)f(x)dx + \int_{\underline{v}}^{\max\{\underline{v}, kv\}} (kv - x)f(x)dx \\
&\quad + \int_v^{\min\{\bar{v}, \frac{v}{k}\}} (v - kx)f(x)dx
\end{aligned}$$

By bidding B , the expected payoff is

$$\begin{aligned}
EU(B, v) &= \int_{\underline{v}}^{\hat{v}} (v - B)f(x)dx + \int_{\underline{v}}^{\max\{\underline{v}, kv\}} (kv - x)f(x)dx \\
&\quad + \int_{\hat{v}}^{\bar{v}} \frac{1}{2}(v - B)f(x)dx + \int_{\hat{v}}^{\max\{\hat{v}, \min\{\bar{v}, \frac{v}{k}\}\}} \frac{1}{2}(v - kx)f(x)dx
\end{aligned}$$

We observe that

$$\begin{aligned}
& EU(NB, v) - EU(B, v) \\
&= \int_{\underline{v}}^v (v - kx)f(x)dx + \int_v^{\min\{\bar{v}, \frac{v}{k}\}} (v - kx)f(x)dx - \int_{\underline{v}}^{\hat{v}} (v - B)f(x)dx \\
&\quad - \int_{\hat{v}}^{\bar{v}} \frac{1}{2}(v - B)f(x)dx - \int_{\hat{v}}^{\max\{\hat{v}, \min\{\bar{v}, \frac{v}{k}\}\}} \frac{1}{2}(v - kx)f(x)dx
\end{aligned} \tag{9}$$

and that this is equal to the negative of (8) when $v = \hat{v}$, i.e. the expression is equal to zero in this case. The derivative of (9) is

$$F(\min\{\bar{v}, \frac{v}{k}\}) - \frac{1}{2} \left(1 + F(\max\{\hat{v}, \min\{\bar{v}, \frac{v}{k}\}\}) \right) < 0$$

implying that $EU(NB, v) - EU(B, v) > 0$ for all $v < \hat{v}$. Thus, low valuation bidders have no incentive to deviate either. This completes the proof of Proposition 2. ■

Proof of Proposition 3. If $EP_2(v|\hat{v})$ denotes the expected payment in stage 2 of a bidder with valuation v when the cut-off valuation is \hat{v} , the expected revenue in stage 2 is

$$ER_2(\hat{v}) = 2 \int_{\underline{v}}^{\bar{v}} EP_2(v|\hat{v}) f(v) dv$$

From the expressions of expected payoff given in the proof of Proposition 2, it follows that

$$EP_2(v|\hat{v}) = \int_{\underline{v}}^{\max\{\underline{v}, kv\}} x f(x) dx + \int_{\underline{v}}^{\min\{\bar{v}, \frac{v}{k}\}} kx f(x) dx$$

for $v < \hat{v}$, and

$$\begin{aligned} EP_2(v|\hat{v}) &= \int_{\underline{v}}^{\min\{\hat{v}, \max\{\underline{v}, kv\}\}} x f(x) dx + \int_{\hat{v}}^{\min\{\bar{v}, \frac{v}{k}\}} \frac{1}{2} kx f(x) dx \\ &\quad + \int_{\hat{v}}^{\max\{\hat{v}, kv\}} \frac{1}{2} x f(x) dx \end{aligned}$$

otherwise. Hence,

$$\begin{aligned} ER_2(\hat{v}) &= 2 \int_{\underline{v}}^{\hat{v}} \left(\int_{\underline{v}}^{\max\{\underline{v}, kv\}} x f(x) dx + \int_{\underline{v}}^{\min\{\bar{v}, \frac{v}{k}\}} kx f(x) dx \right) f(v) dv \\ &\quad + 2 \int_{\hat{v}}^{\bar{v}} \left(\int_{\underline{v}}^{\min\{\hat{v}, \max\{\underline{v}, kv\}\}} x f(x) dx + \int_{\hat{v}}^{\min\{\bar{v}, \frac{v}{k}\}} \frac{1}{2} kx f(x) dx \right. \\ &\quad \left. + \int_{\hat{v}}^{\max\{\hat{v}, kv\}} \frac{1}{2} x f(x) dx \right) f(v) dv \end{aligned}$$

The next step is to change the order of integration of each of the five terms. The first term,

$$T^1 = 2 \int_{\underline{v}}^{\widehat{v}} \int_{\underline{v}}^{\max\{\underline{v}, kv\}} xf(x)f(v)dx dv$$

is obviously zero if $\underline{v} \geq k\widehat{v}$. Otherwise, it is straightforward to change the order of integration to get

$$T_{\underline{v} < k\widehat{v}}^1 = 2 \int_{\underline{v}}^{k\widehat{v}} \int_{\frac{x}{k}}^{\widehat{v}} xf(x)f(v)dv dx$$

Consequently, for any \widehat{v} ,

$$\begin{aligned} T^1 &= 2 \int_{\underline{v}}^{\max\{\underline{v}, k\widehat{v}\}} \int_{\frac{x}{k}}^{\widehat{v}} xf(x)f(v)dv dx \\ &= 2 \int_{\underline{v}}^{\max\{\underline{v}, k\widehat{v}\}} xf(x)(F(\widehat{v}) - F(\frac{x}{k}))dx \end{aligned}$$

Turning to the second term,

$$\begin{aligned} T^2 &= 2 \int_{\underline{v}}^{\widehat{v}} \int_v^{\min\{\overline{v}, \frac{v}{k}\}} kxf(x)f(v)dx dv \\ &= 2 \int_{\underline{v}}^{\min\{k\overline{v}, \widehat{v}\}} \int_v^{\frac{v}{k}} kxf(x)f(v)dx dv + 2 \int_{\min\{k\overline{v}, \widehat{v}\}}^{\widehat{v}} \int_v^{\overline{v}} kxf(x)f(v)dx dv \end{aligned}$$

where the last term is zero if $\widehat{v} < k\overline{v}$. In this case, changing the order of integration yields

$$T_{\widehat{v} < k\overline{v}}^2 = 2 \int_{\underline{v}}^{\frac{v}{k}} \int_{\underline{v}}^{\min\{x, \widehat{v}\}} kxf(x)f(v)dv dx + 2 \int_{\frac{v}{k}}^{\widehat{v}} \int_{kx}^{\min\{x, \widehat{v}\}} kxf(x)f(v)dv dx$$

while for $\widehat{v} \geq k\overline{v}$,

$$T_{\widehat{v} \geq k\overline{v}}^2 = 2 \int_{\underline{v}}^{\frac{v}{k}} \int_{\underline{v}}^{\min\{x, \widehat{v}\}} kxf(x)f(v)dv dx + 2 \int_{\frac{v}{k}}^{\overline{v}} \int_{kx}^{\min\{x, \widehat{v}\}} kxf(x)f(v)dv dx$$

It follows that we can write this term, for all v , as

$$\begin{aligned}
T^2 &= 2 \int_{\underline{v}}^{\frac{\underline{v}}{k}} \int_{\underline{v}}^{\min\{x, \widehat{v}\}} kx f(x) f(v) dv dx \\
&\quad + 2 \int_{\frac{\underline{v}}{k}}^{m(\widehat{v})} \int_{kx}^{\min\{x, \widehat{v}\}} kx f(x) f(v) dv dx \\
&= 2 \int_{\underline{v}}^{m(\widehat{v})} \int_{\max\{\underline{v}, kx\}}^{\min\{x, \widehat{v}\}} kx f(x) f(v) dv dx \\
&= 2 \int_{\underline{v}}^{\widehat{v}} kx f(x) (F(x) - F(\max\{\underline{v}, kx\})) dx \\
&\quad + 2 \int_{\widehat{v}}^{m(\widehat{v})} kx f(x) (F(\widehat{v}) - F(\max\{\underline{v}, kx\})) dx
\end{aligned}$$

Changing the order of integration of the third term, we find that

$$\begin{aligned}
T^3 &= 2 \int_{\underline{v}}^{\max\{\underline{v}, k\widehat{v}\}} \int_{\widehat{v}}^{\overline{v}} x f(x) f(v) dv dx + 2 \int_{\max\{\underline{v}, k\widehat{v}\}}^{\min\{\widehat{v}, k\overline{v}\}} \int_{\frac{x}{k}}^{\overline{v}} x f(x) f(v) dv dx \\
&= 2 \int_{\underline{v}}^{\max\{\underline{v}, k\widehat{v}\}} x f(x) (1 - F(\widehat{v})) dx + 2 \int_{\max\{\underline{v}, k\widehat{v}\}}^{\min\{\widehat{v}, k\overline{v}\}} x f(x) (1 - F(\frac{x}{k})) dx
\end{aligned}$$

The fourth term can be rewritten as

$$\begin{aligned}
T^4 &= \int_{\widehat{v}}^{m(\widehat{v})} \int_{\widehat{v}}^{\overline{v}} kx f(x) f(v) dv dx + \int_{m(\widehat{v})}^{\overline{v}} \int_{kx}^{\overline{v}} kx f(x) f(v) dv dx \\
&= \int_{\widehat{v}}^{m(\widehat{v})} kx f(x) (1 - F(\widehat{v})) dx + \int_{m(\widehat{v})}^{\overline{v}} kx f(x) (1 - F(kx)) dx
\end{aligned}$$

while the fifth and final term is equal to

$$\begin{aligned}
T^5 &= \int_{\min\{\widehat{v}, k\overline{v}\}}^{k\overline{v}} \int_{\frac{x}{k}}^{\overline{v}} x f(x) f(v) dv dx \\
&= \int_{\min\{\widehat{v}, k\overline{v}\}}^{k\overline{v}} x f(x) (1 - F(\frac{x}{k})) dx
\end{aligned}$$

Summing and rearranging the five terms and noting that $\min\{\widehat{v}, k\overline{v}\} = km(\widehat{v})$ produce (6). This ends the proof of Proposition 3. ■

Proof of Lemma 2. (i) The function $m(\hat{v})$ is differentiable everywhere except at $\hat{v} = k\bar{v}$. Hence, for all $\hat{v} \neq k\bar{v}$, the derivative of (6) is

$$\begin{aligned} ER'_2(\hat{v}) &= km(\hat{v})f(km(\hat{v}))(1 - F(m(\hat{v})))m'(\hat{v})k \\ &\quad + \int_{\hat{v}}^{m(\hat{v})} kxf(x)dx f(\hat{v}) - k\hat{v}f(\hat{v})(1 - F(\hat{v})) \\ &\quad + 2km(\hat{v})f(m(\hat{v}))(F(\hat{v}) - F(km(\hat{v})))m'(\hat{v}) \end{aligned}$$

Since the last term is always zero, the derivative can be written as

$$\begin{aligned} &ER'_2(\hat{v}) \\ &= \begin{cases} f(\hat{v}) \left((\hat{v} - k\hat{v})(1 - F(\frac{\hat{v}}{k})) + \int_{\frac{\hat{v}}{k}}^{\frac{\hat{v}}{k}} k(x - \hat{v})f(x)dx \right) & \hat{v} < k\bar{v} \\ f(\hat{v}) \int_{\hat{v}}^{\bar{v}} k(x - \hat{v})f(x)dx & \hat{v} > k\bar{v} \end{cases} \quad (10) \end{aligned}$$

which is strictly positive for all $\hat{v} < \bar{v}$. Note also that when \hat{v} converges to $k\bar{v}$, $ER'_2(\hat{v})$ converges to the same from the left and the right. That is, $ER'_2(\hat{v})$ is continuously differentiable, and strictly increasing.

(ii) Again, the function $m(\hat{v})$ is differentiable everywhere except at $\hat{v} = k\bar{v}$. Thus, for all $\hat{v} \neq k\bar{v}$, the derivative of (5) is

$$\begin{aligned} ER'_1(\hat{v}) &= -f(\hat{v}) \left(\hat{v}(1 - F(m(\hat{v}))) + \int_{\hat{v}}^{m(\hat{v})} kxf(x)dx \right) \\ &\quad + (1 - F(\hat{v}))(1 - F(m(\hat{v})) + k\hat{v}f(\hat{v})) \\ &\quad + (1 - F(\hat{v}))f(m(\hat{v}))m'(\hat{v})(km(\hat{v}) - \hat{v}) \end{aligned}$$

Once more, the last term is always zero. Rewriting yields

$$\begin{aligned} ER'_1(\hat{v}) &= -f(\hat{v}) \int_{\hat{v}}^{m(\hat{v})} k(x - \hat{v})f(x)dx \\ &\quad - f(\hat{v})(1 - F(m(\hat{v}))) \left(\hat{v} - \frac{1 - F(\hat{v})}{f(\hat{v})} - k\hat{v} \right) \end{aligned}$$

or

$$\begin{aligned} &ER'_1(\hat{v}) \\ &= \begin{cases} -f(\hat{v}) \left(\int_{\frac{\hat{v}}{k}}^{\frac{\hat{v}}{k}} k(x - \hat{v})f(x)dx + (1 - F(\frac{\hat{v}}{k})) \left(\hat{v} - \frac{1 - F(\hat{v})}{f(\hat{v})} - k\hat{v} \right) \right) & \hat{v} < k\bar{v} \\ -f(\hat{v}) \int_{\hat{v}}^{\bar{v}} k(x - \hat{v})f(x)dx & \hat{v} > k\bar{v} \end{cases} \quad (11) \end{aligned}$$

As before, when \widehat{v} converges to $k\bar{v}$, $ER'_1(\widehat{v})$ converges to the same from the left and from the right, and it follows that $ER_1(\widehat{v})$ is continuously differentiable. From (10) and (11), we conclude that the derivative of $ER_1(\widehat{v}) + ER_2(\widehat{v})$ is

$$ER'_1(\widehat{v}) + ER'_2(\widehat{v}) = \begin{cases} (1 - F(\frac{1}{k}\widehat{v}))(1 - F(\widehat{v})) > 0 & \widehat{v} < k\bar{v} \\ 0 & \widehat{v} \geq k\bar{v} \end{cases}$$

This completes the proof of Lemma 2. ■

Proof of Proposition 4. Assuming that $\widehat{v} < k\bar{v}$, (5) and (6) imply

$$\begin{aligned} & ER_2(\widehat{v}) - ER_1(\widehat{v}) \\ &= 2 \int_{\underline{v}}^{\widehat{v}} xf(x)(1 - F(\frac{x}{k}))dx + 2 \int_{\underline{v}}^{\widehat{v}} kxf(x) (F(x) - F(\max\{\underline{v}, kx\})) dx \\ &+ 2 \int_{\widehat{v}}^{\frac{\widehat{v}}{k}} kxf(x) (F(\widehat{v}) - F(\max\{\underline{v}, kx\})) dx - 2 \int_{\underline{v}}^{\widehat{v}} kxf(x)(1 - F(x))dx \\ &+ \int_{\frac{\widehat{v}}{k}}^{\bar{v}} kxf(x)(1 - F(kx))dx + \int_{\widehat{v}}^{k\bar{v}} xf(x)(1 - F(\frac{x}{k}))dx \\ &\qquad\qquad\qquad - (1 - F(\widehat{v}))\widehat{v}(1 - F(\frac{\widehat{v}}{k})) \end{aligned}$$

Alternatively, we can write this as

$$ER_2(\widehat{v}) - ER_1(\widehat{v}) = A(\widehat{v}) + B(\widehat{v})$$

where

$$\begin{aligned} & A(\widehat{v}) \\ &= 2 \int_{\underline{v}}^{\widehat{v}} xf(x)(1 - F(\frac{x}{k}))dx + 2 \int_{\underline{v}}^{\widehat{v}} kxf(x) (F(x) - F(\max\{\underline{v}, kx\})) dx \\ &+ 2 \int_{\widehat{v}}^{\frac{\widehat{v}}{k}} kxf(x) (F(\widehat{v}) - F(\max\{\underline{v}, kx\})) dx - 2 \int_{\underline{v}}^{\widehat{v}} kxf(x)(1 - F(x))dx \end{aligned}$$

and

$$\begin{aligned} & B(\widehat{v}) \\ &= \int_{\frac{\widehat{v}}{k}}^{\bar{v}} kxf(x)(1 - F(kx))dx + \int_{\widehat{v}}^{k\bar{v}} xf(x)(1 - F(\frac{x}{k}))dx \\ &\qquad\qquad\qquad - (1 - F(\widehat{v}))\widehat{v}(1 - F(\frac{\widehat{v}}{k})) \end{aligned}$$

Observing that $A(\underline{v}) = 0$ and

$$A'(\hat{v}) = 2f(\hat{v}) \left((\hat{v} - k\hat{v})(1 - F(\frac{\hat{v}}{k})) + \int_{\hat{v}}^{\frac{\hat{v}}{k}} k(x - \hat{v})f(x)dx \right) = 2ER'_2(\hat{v}) > 0$$

we conclude that $A(\hat{v}) > 0$, for all $\hat{v} \in (\underline{v}, k\bar{v}]$. Furthermore, $B(k\bar{v}) = 0$ and

$$B'(\hat{v}) = -(1 - F(\hat{v}))(1 - F(\frac{\hat{v}}{k})) = -(ER'_1(\hat{v}) + ER'_2(\hat{v})) < 0$$

implies that $B(\hat{v}) > 0$, for all $\hat{v} \in [\underline{v}, k\bar{v}]$. It follows that $ER_2(\hat{v}) - ER_1(\hat{v}) > 0$, for all $\hat{v} \in [\underline{v}, k\bar{v}]$. Finally, Lemma 2 ensures that $ER_2(\hat{v}) - ER_1(\hat{v}) > 0$ on $\hat{v} \in (k\bar{v}, \bar{v}]$ as well, since $ER_2(\hat{v})$ increases and $ER_1(\hat{v})$ decreases on this interval. This ends the proof of Proposition 4. ■

Proof of Proposition 5. In the proof of Lemma 2 it was established that $ER_1(\hat{v})$ is continuously differentiable. From (11) we see specifically that

$$ER'_1(\hat{v}) = -f(\hat{v}) \int_{\hat{v}}^{\bar{v}} k(x - \hat{v})f(x)dx, \text{ for } \hat{v} \in [k\bar{v}, \bar{v}]$$

Clearly, this is negative, and strictly so for all $\hat{v} \in [k\bar{v}, \bar{v}]$. It follows that the optimal value of \hat{v} must be strictly lower than $k\bar{v}$. The sequence of auctions is inefficient since a bidder with valuation $\hat{v} < k\bar{v}$ faced by a rival with valuation \bar{v} wins stage 1 with probability 0.5. The efficient outcome in this case is for the bidder with valuation \bar{v} to win both.

However, when $k = 1$, (11) reduces to $(1 - F(\hat{v}))^2 \geq 0$. It follows that when $k = 1$, the optimal value of \hat{v} is \bar{v} . This completes the proof of Proposition 5. ■

Proof of Proposition 6. To prove the second part of Proposition 6, we start with some preliminary remarks.

(i) We first observe that the assumption $\underline{v} \geq kE(v)$ implies that

$$\int_z^{\bar{v}} (z - kx)f(x)dx \geq 0, \forall z \in [\underline{v}, k\bar{v}] \tag{12}$$

To see this, note that the derivative of the function on the left-hand-side with respect to z is

$$f(z) \left[-(1 - k)z + \frac{1 - F(z)}{f(z)} \right]$$

where the term in square brackets is decreasing in z (by Assumption 2). Thus, once the slope becomes negative, it remains negative. Consequently, the function is minimized at one of the end-points. Clearly, the function is positive at $z = k\bar{v}$, while $\underline{v} \geq kE(v)$ ensures that it is non-negative at \underline{v} . Hence, (12) is satisfied.

(ii) Now, consider stage 2. It is easily seen to be a dominant strategy to bid the marginal valuation in stage 2, if the buy-out price was not accepted. Consider a bidder with valuation z , who played his equilibrium strategy in stage 1, but lost. Then, the buy-out price in stage 2 is $B(z)$. To have a discriminating equilibrium, we require that

$$\int_z^{\frac{B(z)}{k}} (z - kx)f(x)dx \geq (z - B(z))\left[F\left(\frac{B(z)}{k}\right) - F(z) + \frac{1}{2}(1 - F\left(\frac{B(z)}{k}\right))\right] \quad (13)$$

In other words, the bidder should prefer rejecting the buy-out price to accepting it. Notice that the right-hand-side can be made arbitrarily small (and the left-hand-side strictly positive) by letting $B(z) \rightarrow z$, implying that there exists $B(\cdot)$ functions such that (13) is indeed satisfied.

(iii) Turn to stage 1. Let $b(v)$ be the candidate for the equilibrium bidding strategy in stage 1, and assume it is strictly increasing. Since the buy-out price is at least \underline{v} , it is convenient to define $B^{-1}(x) = \underline{v}$ if $x \leq \underline{v}$. Then, if a bidder with valuation v decides to bid $b(z)$ in stage 1, his expected payoff is

$$\begin{aligned} EU(z, v) &= \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv), z\}} (kv - B(x))f(x)dx \\ &+ \max \left\{ \int_z^{\max\{z, \min\{\frac{B(z)}{k}, \frac{v}{k}, \bar{v}\}\}} (v - kx)f(x)dx, \right. \\ &\left. (v - B(z))\left[F\left(\min\left\{\frac{B(z)}{k}, \bar{v}\right\}\right) - F(z) + \frac{1}{2}(1 - F\left(\min\left\{\frac{B(z)}{k}, \bar{v}\right\}\right))\right] \right\} \end{aligned} \quad (14)$$

The first term captures the payoffs when the first auction is won. If the bidder won stage 1, it is optimal to accept the buy-out price in stage 2 if and only if it is lower than kv , and this is the second term. We note that if the buy-out price is not accepted, the winner of the first stage is sure to lose in stage 2. Finally, if stage 1 is lost, the bidder may or may not prefer rejecting $B(z)$ to accepting it. Given that the rival follows the equilibrium strategy, this is captured by the third term.

Given these preliminary remarks, we can now show why it is necessary that $b(v)$ takes the form described in Assumption 3. We rule out local deviations first, and then turn to consider non-local deviations.

Local deviations. First, consider $v < k\bar{v}$, and examine the properties of (14) for z close to v . Given (13) is satisfied, and $kv < B(z) < v$ with $z \approx v$, the payoffs in (14) reduce to

$$\begin{aligned} EU(z, v) &= \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ &\quad + \int_z^{\frac{B(z)}{k}} (v - kx)f(x)dx \end{aligned}$$

Taking the derivative with respect to z , it is immediate that the first order condition is satisfied if and only if $b(v)$ is as stated in Assumption 3. Observe further that $b(v) \rightarrow kv$ as $B(v) \rightarrow v$.

Next, consider $v > k\bar{v}$. When $v, z > k\bar{v}$, the buy-out price is $B(z) = z$, which implies that (13) is satisfied. Then, for all $z > k\bar{v}$, the payoffs in (14) reduce to

$$\begin{aligned} EU(z, v) &= \int_{\underline{v}}^{k\bar{v}} (v - b(x))f(x)dx + \int_{k\bar{v}}^z (v - b(x))f(x)dx \\ &\quad + \int_{\underline{v}}^{kv} (kv - B(x))f(x)dx + \int_z^{\bar{v}} (v - kx)f(x)dx \end{aligned}$$

Clearly, this is independent of z if $b(x) = kx$ for all $x \geq k\bar{v}$. Hence, there is no incentive for a bidder with valuation v to bid $b(z)$ rather than $b(v)$ in stage 1. This completes the proof that there is no incentive to make small, local deviations from $b(v)$.

Non-local deviations. Turn to more sizeable deviations. Assume, for now, that $b(v)$ is strictly increasing, and recall that v denotes the true valuation of a bidder, whereas z denotes the valuation the bidder pretends to have by bidding $b(z)$. We rule out the remaining, potential deviations in three steps. The first two deal with upward deviations, while the last covers downward deviations.

(a) $z \geq B(z) > v$. We have already shown that if $v \geq k\bar{v}$, then it does not pay to deviate to a $z = B(z) > v$. Hence, we concentrate on $v < k\bar{v}$, and

observe that it is a dominant strategy in stage 2 not to accept $B(z)$ if stage 1 was lost. Thus, the payoffs in (14) reduce to

$$EU(z, v) = \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ + \int_z^{\max\{z, \frac{v}{k}\}} (v - kx)f(x)dx$$

The derivative with respect to z can be written as

$$EU'_z(z, v) = \begin{cases} (v - kz - (b(z) - kz))f(z) \leq 0 & \text{if } z > \frac{v}{k} \\ (kz - b(z))f(z) \leq 0 & \text{if } z < \frac{v}{k} \end{cases}$$

Thus, deviations of this type are unprofitable, since it is preferable to lower z from any level $z \geq B^{-1}(v) \geq v$.

(b) $z > v \geq B(z)$. This is possible only if $z, v \in (\underline{v}, k\bar{v})$. If a bidder with valuation v loses stage 1 with a bid of $b(z)$, he will elect not to accept $B(z)$ in stage 2 if

$$\int_z^{\frac{B(z)}{k}} (v - kx)f(x)dx \geq (v - B(z))\left[F\left(\frac{B(z)}{k}\right) - F(z) + \frac{1}{2}\left(1 - F\left(\frac{B(z)}{k}\right)\right)\right] \quad (15)$$

Assuming that $B(z)$ is sufficiently close to z to satisfy (13), and noting that the right-hand-side of (15) increases faster in v than the left-hand-side, it follows that the inequality remains satisfied for any $v < z$. The bidder is better off not accepting $B(z)$ in stage 2 if stage 1 was lost. Hence, expected payoff in (14) can be written as

$$EU(z, v) = \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ + \int_z^{\frac{B(z)}{k}} (v - kx)f(x)dx$$

The derivative with respect to z is

$$EU'_z(z, v) = (v - b(z))f(z) + \frac{B'(z)}{k}(v - B(z))f\left(\frac{B(z)}{k}\right) - (v - kz)f(z) \\ = \left[kz + \frac{B'(z)}{k}(v - B(z))\frac{f\left(\frac{B(z)}{k}\right)}{f(z)} - b(z) \right] f(z)$$

But, by the definition of the stage 1 bidding strategy, we have

$$b(z) = kz + \frac{B'(z)}{k}(z - B(z))\frac{f(\frac{B(z)}{k})}{f(z)}$$

and the derivative reduces to

$$EU'_z(z, v) = \frac{B'(z)}{k}(v - z)f(\frac{B(z)}{k}) < 0$$

Thus, this type of deviation is ruled out, as it pays a bidder with valuation v to lower z (hence, $b(z)$) from its high level.

(c) $v \geq z \geq B(z)$. A downward deviation in stage 1 by a bidder with valuation v from $b(v)$ to $b(z)$ clearly increases the probability of losing stage 1. However, if stage 1 was lost, the bidder can choose to either accept or reject $B(z)$ in stage 2. Consider the two options in turn.

(c₁) $v \geq z \geq B(z)$ and *reject* $B(z)$ if stage 1 was lost. The expected payoff in (14) is

$$\begin{aligned} EU(z, v) &= \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv), z\}} (kv - B(x))f(x)dx \\ &\quad + \int_z^{\min\{\frac{B(z)}{k}, \bar{v}\}} (v - kx)f(x)dx \end{aligned}$$

Since the second term is non-decreasing in z , the derivative with respect to z can be bounded below, and we have

$$EU'_z(z, v) \geq \begin{cases} 0 & \text{if } B(z) \geq k\bar{v} \\ (v - z)f(\frac{B(z)}{k})\frac{B'(z)}{k} \geq 0 & \text{if } B(z) < k\bar{v} \end{cases}$$

since $b(z) = kz$ when $B(z) \geq k\bar{v}$. Hence, this type of (downward) deviation is not profitable either.

(c₂) $v \geq z \geq B(z)$ and *accept* $B(z)$ if stage 1 was lost. This type of deviation is a little more tricky than the previous ones, and we approach it in slightly different fashion. First, observe that if $B(z) \geq k\bar{v}$, then the winner of stage 1 will not accept $B(z)$. But then the loser of stage 1 should not accept $B(z)$ either. To see this, we simply note that when $v \geq B(z) \geq k\bar{v}$, the loser of stage 1 is certain to win a second-price auction, and pay less than

$B(z)$. Hence, in order for it to be a sensible strategy for a stage 1 loser with valuation v to accept $B(z)$, we must as a minimum require that $B(z) < k\bar{v}$, or $z < k\bar{v}$. Hence, the payoffs in (14) reduce to

$$\begin{aligned} EU(z, v) &= \int_{\underline{v}}^z (v - b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv), z\}} (kv - B(x))f(x)dx \\ &\quad + (v - B(z))\left[F\left(\frac{B(z)}{k}\right) - F(z) + \frac{1}{2}\left(1 - F\left(\frac{B(z)}{k}\right)\right)\right] \end{aligned}$$

If, in contrast, the bidder with valuation v follows the equilibrium strategy, his payoffs are

$$\begin{aligned} EU(v, v) &= \int_{\underline{v}}^v (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ &\quad + \int_v^{\min\{\frac{B(v)}{k}, \bar{v}\}} (v - kx)f(x)dx \end{aligned}$$

Since

$$\int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \geq \int_{\underline{v}}^{\min\{B^{-1}(kv), z\}} (kv - B(x))f(x)dx$$

it follows that

$$\begin{aligned} EU(v, v) - EU(z, v) &\geq \int_{\underline{v}}^v (v - b(x))f(x)dx + \int_v^{\min\{\frac{B(v)}{k}, \bar{v}\}} (v - kx)f(x)dx \\ &\quad - \int_{\underline{v}}^z (v - b(x))f(x)dx \\ &\quad - (v - B(z))\left[F\left(\frac{B(z)}{k}\right) - F(z) + \frac{1}{2}\left(1 - F\left(\frac{B(z)}{k}\right)\right)\right] \\ &= \int_z^v (v - b(x))f(x)dx + \int_v^{\min\{\frac{B(v)}{k}, \bar{v}\}} (v - kx)f(x)dx \\ &\quad - (v - B(z))\left[F\left(\frac{B(z)}{k}\right) - F(z) + \frac{1}{2}\left(1 - F\left(\frac{B(z)}{k}\right)\right)\right] \\ &= D(z, v) \end{aligned}$$

Hence, deviations of the kind considered are ruled out, if we can show that $D(z, v) \geq 0$.

If $v \geq k\bar{v}$, the facts that $B(v) \geq k\bar{v}$ and $b(x) = kx$ for $x \geq k\bar{v}$ imply that $D(z, v)$ reduces to

$$D(z, v) = \int_z^{\bar{v}} (B(z) - b(x))f(x)dx + (v - B(z))\frac{1}{2}(1 - F(\frac{B(z)}{k}))$$

Since the last term is positive and the first converges to (12) for $B(x) \rightarrow x$, it follows that $D(z, v) > 0$ for $B(\cdot)$ functions that are sufficiently close to the 45° line.

Finally, if $v < k\bar{v}$, then $D(z, v)$ is positive for $z = v$ since (13) must be satisfied (that is, it must be optimal to reject the buy-out price in the putative equilibrium). We wish to show that $D(z, v)$ is also positive for $z < v$. So, differentiate $D(z, v)$ with respect to v to obtain

$$D'_v(z, v) = F(\frac{B(v)}{k}) - \frac{1}{2}(1 + F(\frac{B(z)}{k}))$$

Observing that $D'_v(z, z) < 0$, $D'_v(z, k\bar{v}) > 0$ and $D''_{vv}(z, v) > 0$, it follows that the minimum of $D(z, v)$ over $v \in (z, k\bar{v})$ is interior, and satisfies

$$F(\frac{B(v)}{k}) = \frac{1}{2}(1 + F(\frac{B(z)}{k}))$$

Hence, while noting that $\min\{\frac{B(v)}{k}, \bar{v}\} = \frac{B(v)}{k}$, we conclude that

$$\begin{aligned} D(z, v) &= \int_z^v (v - b(x))f(x)dx + \int_v^{\frac{B(v)}{k}} (v - kx)f(x)dx \\ &\quad - (v - B(z))[F(\frac{B(z)}{k}) - F(z) + \frac{1}{2}(1 - F(\frac{B(z)}{k}))] \\ &= v[F(\frac{B(v)}{k}) - \frac{1}{2}(1 + F(\frac{B(z)}{k}))] + B(z)[\frac{1}{2}(1 + F(\frac{B(z)}{k})) - F(z)] \\ &\quad - \int_z^v b(x)f(x)dx - \int_v^{\frac{B(v)}{k}} kxf(x)dx \\ &\geq B(z)[F(\frac{B(v)}{k}) - F(z)] - \int_z^v b(x)f(x)dx - \int_v^{\frac{B(v)}{k}} kxf(x)dx \\ &= \int_z^v (B(z) - b(x))f(x)dx + \int_v^{\frac{B(v)}{k}} (B(z) - kx)f(x)dx \\ &> \int_z^{k\bar{v}} (B(z) - b(x))f(x)dx + \int_{k\bar{v}}^{\bar{v}} (B(z) - kx)f(x)dx \end{aligned}$$

where the last inequality follows from the facts the function preceding it is decreasing in v and $v < k\bar{v}$. As $B(x) \rightarrow x$, this converges to (12), and, again, we conclude that $D(z, v)$ is positive for $B(\cdot)$ functions sufficiently close to the 45° line.

Hence, we conclude that if (12) is satisfied, there exists a $B(\cdot)$ function close to the 45° line, for which there is no incentive to deviate, regardless of the bidder's valuation.

It remains only to verify that the stage 1 bidding strategy, $b(v)$, is strictly increasing. However, it is clear that for $B(v) \rightarrow v$ (with $B'(v) < \infty$) this must be the case as $b(v) \rightarrow kv$. Since $kJ(v) > J(kv)$, it follows that the second part of Assumption 3 is satisfied as well, for $B(v) \rightarrow v$. This completes the proof of Proposition 6. ■

Appendix B

In this appendix we show that all results of Section 3 hold with minor modifications when Assumption 1 is not met.

Observe first that Proposition 2 and $ER_1(\hat{v})$ in Proposition 3 hold even when Assumption 1 is not satisfied. Consequently, the derivative of $ER_1(\hat{v})$ is

$$\begin{aligned} ER_1'(\hat{v}) &= -f(\hat{v}) \int_{\hat{v}}^{m(\hat{v})} k(x - \hat{v})f(x)dx \\ &\quad - f(\hat{v})(1 - F(m(\hat{v}))) \left(\hat{v} - \frac{1 - F(\hat{v})}{f(\hat{v})} - k\hat{v} \right) \\ &= -f(\hat{v}) \int_{\hat{v}}^{\bar{v}} k(x - \hat{v})f(x)dx \leq 0 \end{aligned}$$

since $m(\hat{v}) = \bar{v}$. This immediately implies that the optimal value of \hat{v} is \underline{v} , and the buy-out price is thus accepted with probability 1.

Furthermore, since $k\bar{v} \leq \underline{v}$, it is clear that whoever loses stage 1 will win stage 2 with probability 1, regardless of \hat{v} . Hence, by the *Revenue Equivalence Theorem*, overall revenue is the same⁴³ regardless of \hat{v} . Since $ER_1(\hat{v})$ is decreasing in \hat{v} , it follows that $ER_2(\hat{v})$ is increasing in \hat{v} (the equivalent of Lemma 2).

In addition, since the optimal value of \hat{v} is \underline{v} , the highest possible revenue to the first seller is $ER_1(\underline{v}) = B(\underline{v}) = kE(v)$. In stage 2, the loser of stage 1 will win. Defining $v_{(j)}$ as the j 'th highest valuation, the expected revenue is $ER_2(\underline{v}) = 0.5kE(v_{(1)}) + 0.5kE(v_{(2)})$, since any given player wins stage 1 with probability 0.5. This can be rewritten as

$$\begin{aligned} ER_2(\underline{v}) &= 0.5kE(v_{(1)}) + 0.5kE(v_{(2)}) \\ &= 0.5kE(v_{(1)}) + 0.5k(2E(v) - E(v_{(1)})) \\ &= kE(v) \\ &= ER_1(\underline{v}) \end{aligned}$$

Hence, in what seller 1 considers optimum, he earns the same as seller 2. Since the sum of revenues is constant, it follows that for any $\hat{v} > \underline{v}$, seller 1 will be worse off than seller 2, and we have the equivalent of Proposition 4. ■

⁴³It is easily seen that an agent of type \underline{v} is indifferent between the auction formats.