Ranking Reversals in Asymmetric Auctions*

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Abstract

This paper compares the first-price auction and the second-price auction with several asymmetric bidders who are either weak or strong. The ranking of these auctions in terms of profit may flip as the exogenous reserve price or the number of weak or strong bidders change. Similarly, with endogenous reserve prices the ranking may depend on the seller's own-use valuation. In other words, the ranking may be fragile to changes along these dimensions. Existing models rule out such ranking reversals by imposing substantial structure on type distributions. The current paper relies on simple mechanism design arguments that require less structure.

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1 Introduction

Starting with Vickrey's (1961) seminal work, a central preoccupation of auction theory has been to rank the efficiency and profitability of different types of auctions. Vickrey (1961) himself was first to identify asymmetry among bidders as a critical factor. To this day, asymmetric auctions remain less than perfectly understood.

Vickrey (1961) demonstrated that there is no unambiguous profit ranking of the first-price auction (FPA) and the second-price auction (SPA). In a well-known paper, Maskin and Riley (2000) developed a few general principles for when the FPA outperforms the SPA, and vice versa. However, they concentrated on auctions with two bidders, one of whom is ex ante perceived as "strong" and the other as "weak". Kirkegaard (2012a) generalized Maskin and Riley's (2000) insights using a mechanism design approach. This literature centres on the role of bidders' type distributions. Simply put, the lesson is that the profit ranking depends on the type distributions.[1]

However, the type distributions need not be the only consideration for the seller when she contemplates different auction formats. How many weak and strong bidders are at auction? Is the reserve price predetermined by e.g. government regulation and, if so, at what level? On the other hand, if the reserve price is under the seller's control then she must, in order to determine the optimal reserve price, ask herself what her own-use value of the object is in case it is not sold? Unlike previous papers, the current paper focuses on the role of these *parameters* of the problem. Given the type distributions, the primary objective is to examine whether the ranking of asymmetric auctions is generally robust to changes in the parameters. This is essentially the reverse of the exercise in Maskin and Riley (2000). They hold fixed the parameters and ask whether the distributions matter.

As mentioned, Maskin and Riley (2000) assume there are exactly two bidders. They also ignore reserve prices. Maskin and Riley (2000) and Kirkegaard (2012a) set out to identify configurations of type distributions where auctions can be ranked.[2] This necessitates the use of fairly demanding proof techniques. As noted by Kirkegaard (2012a), these models are therefore heavily structured, and in fact so much so that

[1] This is a recurrent theme in auction theory. For a recent example, see Baisa and Burkett (2018) who compare uniform-price auctions and discriminatory auctions with one large bidder and many small bidders. See also Baisa and Burkett (2020).

[2] This is exemplified by Maskin and Riley's (2000) "shift" and "stretch" models. These models are special cases of those identified in Kirkegaard (2012a). Cheng (2006) describes another configuration of distributions in which the FPA is superior.

the profit ranking is unchanged when reserve prices are allowed and when more weak bidders are present.[3]

This paper does not impose such rigid structure on the configuration of type distributions. Thus, it will be shown that there are configurations where the profit ranking is sensitive to exogenous changes in reserve prices and the numbers of weak and strong bidders. Second, endogenizing the reserve price and recognizing that it depends on the seller's own-use valuation likewise leads to the conclusion that the ranking may also depend on the latter parameter.[4]

The research question is motivated by a large empirical literature on asymmetric auctions. This literature generally lumps bidders into two groups. Campo, Perrigne, and Vuong (2003) divide bidders into solo bidders and joint bidders. In De Silva, Dunne, and Kosmopoulou (2003) bidders are either entrants or incumbents. The bidders in Flambard and Perrigne's (2006) study are located in one of two areas. Brendstrup and Paarsch (2006) consider an application with major and minor bidders. Likewise, in Marion (2007) and Krasnokutskaya and Seim (2011) bidders are classified as either large or small. Finally, Athey, Levin, and Seira (2011) put loggers and sawmills in separate groups.[5]

Given the sparsity of theoretical results, the empirical literature is forced to resort to numerical analysis in order to compare the performance of the observed auction format to that of some counterfactual auction format. However, it is unclear how robust the empirical findings are to changes in the parameters of the problem. The practical significance of the current paper is to highlight that the design question should be revisited following any change in parameters. For instance, when bidders are firms in the same industry, a change in the industry structure may make it optimal for the seller to switch auction format. Likewise, if the government's opportunity cost of timber is to use the forest as a carbon sink instead, increased environmental awareness may cause the optimal reserve price to change in timber auctions, and with it the ranking of different auctions formats.

[3] Maskin and Riley's (2000) example in which the SPA is more profitable than the FPA is also robust to an arbitrary number of strong bidders. Kirkegaard (2012b) contains two examples in which the FPA outperforms the SPA for an arbitrary number of weak and strong bidders.

[4] A related point is made by Doni and Menicucci (2013) in a two-bidder model with binary types. They find that the FPA may be superior to the SPA without a reserve price. However, with optimal reserve prices the SPA is weakly better than the FPA in their model.

[5] Even if bidders are ex ante symmetric, collusion among a subset of bidders effectively create asymmetries. See e.g Asker (2010) for a study of collusion in stamp auctions.

Motivated by the above literature, this paper considers auctions with two groups of bidders. Bidders in one group are strong compared to bidders in the other group who are weak. Formally, this is captured by the standard assumption that one type distribution dominates the other in terms of the reverse hazard rate.

There are two key assumptions that work in tandem to simplify the analysis. First, it is assumed that there are at least two strong bidders at auction. This is in contrast to much of the existing theoretical literature, such as Maskin and Riley (2000) and Kirkegaard (2012a). In most of the auctions considered in the empirical literature, however, there are more than one strong bidder present at auction. Hence, the assumption in the current paper is empirically justified.

The second key assumption is that the two type distributions do not have the same support. The highest type of a strong bidder strictly exceeds the highest type of a weak bidder. Then, there may exist a range of high bids that are only ever submitted by strong bidders in equilibrium. The reason is that competition among the strong bidders entice them to bid so aggressively that weak bidders cannot keep up. This phenomenon is referred to as *bid-separation*.[6] Note that bid-separation never occurs with only one strong bidder. Although bid-separation may seem to complicate the problem, the opposite is in fact that case. The core methodological insight is that bid-separation may make it possible to apply elementary mechanism design techniques that actually fail when there is only one strong bidder.

Following Myerson (1981), it is well understood that one auction is more profitable than another if it allocates the object to a bidder with a weakly higher "virtual valuation" with probability one. It is this basic mechanism design result that will be utilized here. This particular part of the approach is not claimed to be novel. Nevertheless, as emphasized by Maskin and Riley (2000), the argument does not in general have enough bite to compare the SPA and FPA. Thus, the novelty comes from identifying an instrument that can be leveraged to invoke the simple argument. To this end, the first observation is that the argument applies if bid-separation is sufficiently pronounced. Then, strong bidders with high virtual valuations separate away from weak bidders with mediocre virtual valuations. Thus, bid-separation may constitute an opening into the problem, yet the degree of bid-separation is endogenous.

Thus, the second and key observation is that the incidence of bid-separation de-

[6] Note that bid-separation does not refer to the obvious property that bidding strategies are different across bidders in the FPA. Rather, it refers to differences in the ranges of equilibrium bids.

pends on the size of the reserve price. In other words, the reserve price represents a lever that can generate enough bid-separation in equilibrium to permit the use of the fundamental mechanism design argument outlined above. Thus, a ranking can be obtained for some, but not necessarily all, reserve prices.

Evidently, reserve prices play a pivotal role. To start, assume that the reserve price is exogenous and the same for both auctions. Then, for *any* configuration of type distributions that satisfies the model's sparse structure, there always exist *some* reserve price for which the FPA is strictly more profitable than the SPA. On the other hand, it is easy to show that there are configurations where the SPA outperforms the FPA for a subset of reserve prices. Hence, there are configurations in which the profit ranking changes as the reserve price changes. The ranking may likewise change as more bidders join the auction. A concrete example exhibiting these reversal properties is provided. This appears to be the first such example in the literature.

When it is endogenous, the optimal reserve price depends on the type distributions and the parameters as well as the auction format itself. The optimal reserve price in either auction is low enough to permit the weak bidders a chance of winning when neither the asymmetry between distributions or the seller's own-use valuation is too large. Then, the FPA with an optimal reserve price is strictly more profitable than the SPA with an optimal reserve price when there are sufficiently many bidders. Similarly, for a fixed set of bidders, there are own-use valuations for which the FPA outperforms the SPA when the reserve price is endogenized. This latter result is used to show in another example that the profit ranking may flip as the seller's own-use valuation changes. Together, the two examples thus demonstrate the economically important point that profit rankings may be sensitive to the changes in parameters.

2 Model

Two groups of risk neutral bidders participate in a FPA or SPA. Bidders in the strong group independently draw a valuation from the twice continuously differentiable distribution function $F_s(v)$, with support $[\underline{v}_s, \overline{v}_s]$. The density, $f_s(v)$, is assumed to be strictly positive for all $v \in (\underline{v}_s, \overline{v}_s]$. Note that mass points are ruled out. There are a total of $m_s \geq 2$ strong bidders. There are also $m_w \geq 1$ weak bidders. These bidders independently draw a valuation from another twice continuously differentiable distribution function $F_w(v)$, $v \in [\underline{v}_w, \overline{v}_w]$. Again, it is assumed that the density $f_w(v)$

is strictly positive for all $v \in (\underline{v}_w, \overline{v}_w]$. Assume that $\overline{v}_s > \overline{v}_w > \underline{v}_s \geq \underline{v}_w$. Thus, the supports overlap. Note that $\underline{v}_s$ and $\underline{v}_w$ may or may not coincide. However, the strict inequality $\overline{v}_s > \overline{v}_w$ is crucial to the key arguments.

Finally, it is assumed that F_s dominates F_w in terms of the reverse hazard rate, $F_w \leq_{rh} F_s$, or

$$\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)} \text{ for all } v \in (\underline{v}_s, \overline{v}_w]. \tag{1}$$

In other words, $\frac{F_s(v)}{F_w(v)}$ is non-decreasing on $(\underline{v}_s, \overline{v}_w]$. Hence, a strict version of first order stochastic dominance applies since $F_s(v) < F_w(v)$ for all $v \in (\underline{v}_s, \overline{v}_w]$. A generic member of the strong (weak) group is for simplicity referred to as bidder s (w). The number and composition of bidders, i.e. m_s and m_w, are exogenous.

Together, the assumptions that there only two groups of bidders and that $\overline{v}_s > \overline{v}_w$ and $F_w \leq_{rh} F_s$ may seem restrictive. However, many of the paper's results are "negative" results that demonstrate that "robust" rankings are generally not possible. The assumptions just mentioned are sufficient to establish this point. Moreover, as Li and Riley (2007) argue, equilibrium behavior in a model with $\overline{v}_s > \overline{v}_w$ is approximately the same as in a model in which the support of F_w is extended to $[\underline{v}_w, \overline{v}_s]$ but where $f_w(v)$ is near-zero on $(\overline{v}_w, \overline{v}_s]$. Lebrun (2006) employs such an extension, but he allows the density to be exactly zero.

The seller is also risk neutral. Her own-use valuation is denoted z. Thus, z describes the utility that she obtains if the good is not sold, which may be the case if the reserve price, r, is larger than $\underline{v}_s$. Here, z is exogenous. The reserve price may be fixed or exogenous if the auction is subject to some government regulation but the more interesting case is when the reserve price is endogenous and determined by the seller. Both possibilities are analyzed, starting with the former. When r is exogenous and the same across auctions, z is irrelevant for ranking the SPA and the FPA. In this case, z is thus sometimes omitted from the description of the problem.

Following Cantillon (2008), the pair (F_s, F_w) will be referred to as the configuration of type distributions. In contrast, m_s, m_w, z, and – when it is exogenous – also r are *parameters* of the problem. Together, the configuration of type distributions and the parameters define the auction *setting*. Holding fixed the type distributions, the main objective of the paper is to explore the robustness of the profit ranking to changes in the parameters. It is already known from Vickrey (1961) and Maskin and Riley

(2000) that the profit ranking may change when the type distributions change.[7]

Note that competition between strong bidders means that the price cannot fall below $\underline{v}_s$. Hence, a reserve price below $\underline{v}_s$ has no bite. Thus, it is without loss of generality to consider reserve prices, r, that is no smaller than $\underline{v}_s$, or $r \geq \underline{v}_s$, where the special case that $r = \underline{v}_s$ is equivalent to the absence of a reserve price. The reserve price is non-trivial when $r > \underline{v}_s$, in which case the good will remain unsold with positive probability. Reserve prices above $\overline{v}_w$ effectively exclude weak bidders from the auction. Thus, only strong bidders are active. Since active bidders are symmetric, it follows from the Revenue Equivalence Theorem that the SPA and the FPA are equally profitable. Hence, reserve prices in the range $[\underline{v}_s, \overline{v}_w)$ are more interesting. However, the case where $r = \underline{v}_s$ presents some technical difficulties. Consequently, some of the following results focus on exogenous reserve prices in $(\underline{v}_s, \overline{v}_w)$.

3 Equilibrium and comparative statics of the FPA

Lebrun (2006) characterizes equilibrium in the FPA under more general assumptions than those stated above. He proves equilibrium is unique whenever $r > \underline{v}_s$. For any $r \in [\underline{v}_s, \overline{v}_w]$, bidder i with type $v \geq r$ submits a bid in the interval $[r, \overline{b}_i]$, $i = s, w$. Naturally, $\overline{b}_s$ and $\overline{b}_w$ are endogenously determined, with $\overline{b}_s \geq \overline{b}_w$. Thus, all bidders submit bids in the same range if and only if $\overline{b}_s = \overline{b}_w$. This is the case in Maskin and Riley (2000) where $m_s = m_w = 1$. If $\overline{b}_s > \overline{b}_w$ in such a setting, the lone strong bidder who is supposed to bid $\overline{b}_s$ would profit by slightly lowering his bid since it would not affect his chances of winning. That argument breaks down as soon as $m_s \geq 2$, as in the current paper. In fact, the central arguments of the paper rely on the possibility that $\overline{b}_s > \overline{b}_w$. The term bid-separation is henceforth used to refer to any equilibrium in which $\overline{b}_s > \overline{b}_w$. The term is justified by an analogy to signaling games: Any outside observer who does not a priori know the identity of bidders will be able to infer that any given bidder is strong if he submits a "separating" bid in the interval $(\overline{b}_w, \overline{b}_s]$.

In equilibrium, there exists a unique threshold type, $\widehat{v}$, such that bidder s with type $\widehat{v}$ bids $\overline{b}_w$ in the FPA. Higher types separate away from weak bidders by bidding above

[7] An increase in r is similar to a change in the type distributions that leads more probability mass to be concentrated at r. Even though it is known that the profit ranking is sensitive to certain kinds of changes in type distributions, this does not on its own imply that the ranking is sensitive to changes in r. After all, a change in r is a very specific kind of change. Similarly, a change in m_s or m_w means that the distribution of the highest rival type changes in a very specific way.

$\overline{b}_w$. In contrast, types below $\widehat{v}$ engage with weak bidders and may thus potentially lose to a weak bidder. Note that bid-separation takes place if and only if $\widehat{v} < \overline{v}_s$. It follows from Lebrun's (2006) equilibrium characterization that

$$\widehat{v} = \min\left\{\overline{v}_s, \frac{m_s}{m_s-1}\overline{v}_w - \frac{1}{m_s-1}\overline{b}_w\right\}.^8 \tag{2}$$

Now, since $\overline{b}_w$ is bounded between r and $\overline{v}_w$, the above relationship proves formally that $\widehat{v}$ converges to $\overline{v}_w$ as r converges to $\overline{v}_w$. In other words, bid-separation must occur when the reserve price is high enough and $m_s \geq 2$. Note also that $\widehat{v} > \overline{v}_w$ for any $r < \overline{v}_w$. Thus, a weak bidder with type $\overline{v}_w$ outbids strong bidders with higher types. In other words, he wins more often than is efficient.

Given some endogenous $(\widehat{v}, \overline{b}_w)$, the challenge is to describe equilibrium behavior at bids between r and $\overline{b}_w$, or characterizing the interaction between weak and strong bidders with types below $\overline{v}_w$ and $\widehat{v}$, respectively. At bids above $\overline{b}_w$, the auction is essentially a symmetric auction since only strong bidders have types that are active there. Given $(\widehat{v}, \overline{b}_w)$, it is thus trivial to describe the bidding behavior of strong bidders with types above $\widehat{v}$; see Lebrun (2006) and Hubbard and Kirkegaard (2019).

Lebrun (2006) and Hubbard and Kirkegaard (2019) characterize equilibrium of the FPA by describing inverse bidding strategies. However, from a mechanism design perspective it is often more fruitful to characterize the equilibrium allocation instead. Thus, as in Kirkegaard (2012a), the problem is reformulated. Consider a weak bidder with type $v \geq r$. Let $b_w(v)$ denote his equilibrium bid. In equilibrium, this bid equals the bid submitted by a strong bidder with some type $k(v)$.[9] Hence, the weak bidder wins if and only if all the other weak bidders have types below v and all the strong bidders have types below $k(v)$. For bids below $\overline{b}_w$, equilibrium can thus be characterized by describing the pair of (b_w, k) functions instead of the pair of inverse bidding functions. In either case, the endogenous functions solve a system of differential equations with appropriate boundary conditions and initial conditions. In the formulation used here, the boundary conditions are that $k(\overline{v}_w) = \widehat{v}$ and $b_w(\overline{v}_w) = \overline{b}_w$. The initial conditions are described later. The relevant system of differential

[8] See Hubbard and Kirkegaard (2019) for a more detailed discussion of bid-separation. They assume bidders belong to one of two groups, without assuming reverse hazard rate dominance. They also present several comparative statics results. However, they hold the reserve price fixed.

[9] Following Lebrun (2006), $k(v)$ is unique when $r > \underline{v}_s$. There may be multiple equilibria when $r = \underline{v}_s$ but the results hold regardless of which equilibrium is selected in this case.

equations is described in the beginning of Appendix A.

Equilibrium depends on the parameters (r, m_s, m_w) but not on the parameter z which is irrelevant to bidders. Thus, I generally write the endogenous functions as $k(v|r, m_s, m_w)$ and $b_w(v|r, m_s, m_w)$, respectively, but make use of the shorter form $k(v)$ and $b_w(v)$ whenever no confusion arises as a result.

Note that a weak bidder with type v bids more aggressively than a strong bidder with type v if and only if $k(v) > v$. Recall that $k(\overline{v}_w) = \widehat{v} > \overline{v}_w$ when $r < \overline{v}_w$. Indeed, it is a standard result that reverse hazard rate dominance implies $k(v) > v$ globally; see e.g. Lebrun (1999) and Maskin and Riley (2000) for proofs of this result in various settings. The following lemma proves that the property holds in the present setting.

Lemma 1 *Assume $r \in [\underline{v}_s, \overline{v}_w)$. Then, $k(v) > v$ for all $v \in (r, \overline{v}_w]$.*

Proof. See Appendix A. ■

In order to examine the robustness of auction rankings with respect to the parameters of the problem, it is necessary to first understand how the allocation, or $k(v|r, m_s, m_w)$, depends on these. Consider first changes in the reserve price. The easiest way to see that the allocation must change is to note that the "initial conditions" to the system of differential equations change. In particular, combining Lemma 1 and Lebrun's (2006) analysis implies that $b_w(r) = r$ and $k(r) = r$, as explained in the proof of Proposition 1. The first comparative statics result is a monotonicity result. Specifically, $k(v)$ is decreasing in r as illustrated in the left panel of Figure 1.

Proposition 1 *Assume $m_s \geq 2$, $m_w \geq 1$. If $\overline{v}_w > r' > r \geq \underline{v}_s$ then*

$$k(v|r', m_s, m_w) < k(v|r, m_s, m_w) \text{ for all } v \in [r', \overline{v}_w).$$

Proof. See Appendix A. ■

Consider a weak bidder with some type $v \in [r', \overline{v}_w)$. When the reserve price increases from r to r', this bidder becomes less likely (Proposition 1) to outbid the strong bidders and win the FPA. However, it is still the case that he wins more often than is efficient (Lemma 1). Increasing the number of bidders has a similar effect.

Proposition 2 *Assume $m_s' > m_s \geq 2$, $m_w' > m_w \geq 1$, $r \in (\underline{v}_s, \overline{v}_w)$, and $\overline{v}_s > \widehat{v} = k(\overline{v}_w|r, m_s, m_w)$. Then,*

$$k(v|r, m_s', m_w) < k(v|r, m_s, m_w) \text{ and } k(v|r, m, m_w') < k(v|r, m_s, m_w) \text{ for all } v \in (r, \overline{v}_w).$$

Proof. See Appendix A. ∎

The right panel of Figure 1 illustrates Proposition 2. This can also be thought of as a monotonicity result. In particular, the auction becomes closer and closer to efficient the more bidders are participating in the auction. Any bidder responds to the competition he faces. This competition consists of all the remaining bidders in his own group and all the bidders in the other group. As the group sizes grow, however, the difference between the competition faced by a strong and a weak bidder diminishes. Therefore, bidding strategies become more similar.

This result complements Swinkels' (1999, 2001) finding that the FPA is asymptotically efficient. In other words, $k(v|r, m_s, m_w)$ converges to v as the number of bidders goes to infinity. The implication that $k(v|r, m_s, m_w)$ is not bounded away from v in the limit is useful. For completeness, the next result states and proves this fact.

Proposition 3 *Assume $m_s \geq 2$, $m_w \geq 1$, and $r \in [\underline{v}_s, \overline{v}_w)$. Then $k(v|r, m_s, m_w) \to v$ for all $v \in (r, \overline{v}_w]$ as $m_s \to \infty$ or $m_w \to \infty$.*

Proof. See Appendix A. ∎

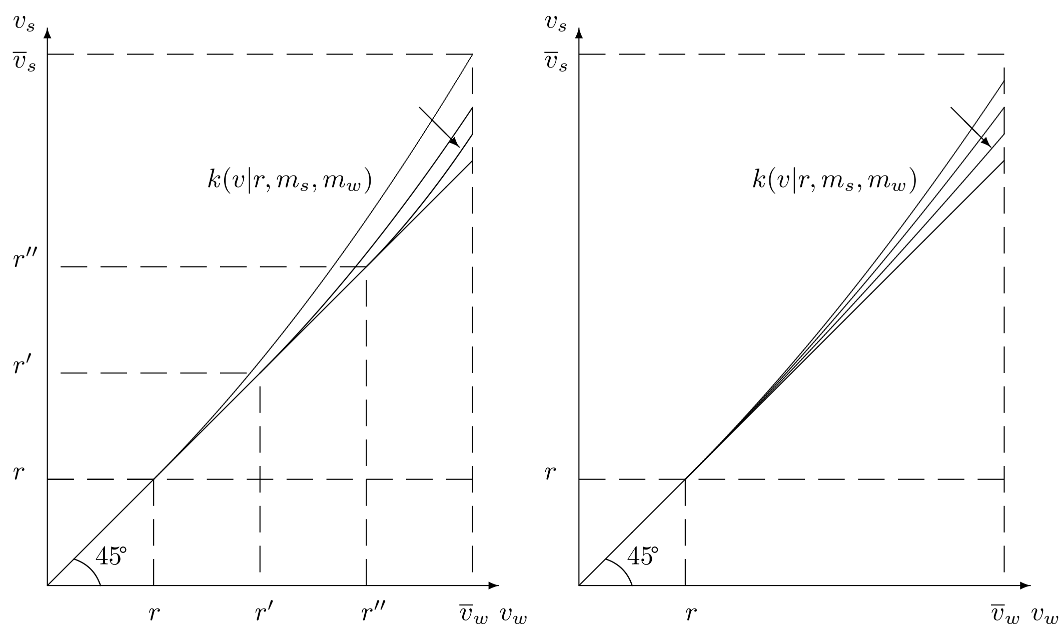

Figure 1: (a) The left panel depicts how $k(v|r, m_s, m_w)$ changes with r, given (m_s, m_w); (b) The right panel shows how $k(v|r, m_s, m_w)$ changes with (m_s, m_w).

4 Ranking auctions for a subset of parameters

Building on Propositions 1–3, the purpose of this section is to rank the FPA and SPA in terms of profit for a subset of the parameters r, m_s, and m_w. Specifically, for all configurations of type distributions that satisfy the model's assumptions, the FPA yields higher expected profit than the SPA if the reserve price is relatively high or if the number of bidders is large. These results will be used in later sections to show that a reversal of the profit ranking is possible for some configurations.

4.1 A method to rank auctions

Myerson (1981) shows that expected revenue in any auction equals the expected value of the winner's virtual valuation.[10] Bidder i's virtual valuation is

$$J_i(v) = v - \frac{1 - F_i(v)}{f_i(v)}.$$

The comparative statics in the previous section are useful because they reveal how the allocation in the FPA depends on the reserve price and the composition of bidders. Let $ER^{FPA}(r, m_s, m_w)$ denote the expected revenue in the FPA given (r, m_s, m_w). However, the seller cares about more than expected revenue since she earns utility of z if the object is not sold. Thus, her expected payoff in the FPA is

$$\Pi^{FPA}(z, r, m_s, m_w) = zF_s(r)^{m_s} F_w(r)^{m_w} + ER^{FPA}(r, m_s, m_w).$$

The literature often assumes that $z = 0$. This is an innocent normalization if the reserve price is exogenous. However, when the reserve price is endogenous, its optimal value typically depends on z. Optimal reserve prices are examined Section 7. For now, the reserve price is thought of as exogenous and the same across auctions. In this case, the auctions are ranked the same way in terms of revenue and profit. More generally, however, it is the profit ranking that matters to the seller.

Although there are multiple equilibria in the SPA, I focus on the equilibrium in which bidders use the weakly dominant strategy of bidding truthfully. When it is sold, the good is thus allocated to the bidder with the highest type. Let $ER^{SPA}(r, m_s, m_w)$

[10] This statement is true whenever any bidder earns zero payoff when he has the lowest possible type in his type support. That property holds here due to the assumptions that $m_s \geq 2$ and $\underline{v}_s \geq \underline{v}_w$.

and $\Pi^{SPA}(z, r, m_s, m_w)$ denote the expected revenue and the expected payoff or profit to the seller in the SPA, respectively.

Recall that the weak bidder with the highest valuation wins more often in the FPA than in the SPA, since $k(v) > v$ for $v > r$. Hence, as noted by Kirkegaard (2012a), for a fixed reserve price, $r \in [\underline{v}_s, \overline{v}_w]$, the difference in profit between the two auctions is

$$\begin{aligned} \Delta(r, m_s, m_w) &= \Pi^{FPA}(z, r, m_s, m_w) - \Pi^{SPA}(z, r, m_s, m_w) \\ &= \int_r^{\overline{v}_w} \left(\int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x))\, dF_s(x)^{m_s} \right) dF_w(v)^{m_w}. \end{aligned} \quad (3)$$

Intuitively, the inner integral in (3) captures the fact that when the most competitive of the weak bidders wins in the FPA but not in the SPA it is because the most competitive bidder in the strong group has a type above v but below $k(v)$. As mentioned earlier, if the reserve price is so high that weak bidders are excluded, or $r \geq \overline{v}_w$, then the two auctions allocate the object in the same way. In this case the SPA and FPA are equally profitable, or $\Delta(r, m_s, m_w) = 0$. Thus, in the remainder of this section and the next two, it is assumed that $r \in [\underline{v}_s, \overline{v}_w)$. The possibility that $r \geq \overline{v}_w$ arises later when endogenous reserve prices are considered.

Expected profit in the FPA is strictly higher than expected profit in the SPA if the parameters (r, m_s, m_w) belong to the set

$$\mathcal{P} = \{(r, m_s, m_w) | J_w(v) - J_s(x) > 0 \text{ for all } x \in [v, k(v|r, m_s, m_w)] \text{ and all } v \in (r, \overline{v}_w]\},$$

in which case each term in the inner integral in (3) is strictly positive. In this case, when the allocation in the FPA differs from the allocation in the SPA it is because the item has been awarded to a bidder with a strictly higher virtual valuation.

However, Maskin and Riley (2000) point out that

$$J_s(\overline{v}_s) > J_w(\overline{v}_w) > J_s(\overline{v}_w). \quad (4)$$

Hence, from a profit perspective it is desirable that the weak bidder with type $\overline{v}_w$ wins more often than is efficient. However, he wins too often if he outbids strong bidders with types close to $\overline{v}_s$. Note that this must necessarily occur if there is no bid-separation, as is the case in any two-bidder model or more generally if $m_s = 1$.

Stated differently, there is no $(r, m_s, m_w) \in \mathcal{P}$ for which $m_s = 1$. Thus, Maskin and Riley (2000) conclude that "mechanism design considerations do not settle the matter of which auction generates more revenue." The innovation in Kirkegaard (2012a) is based on the observation that what is important is not whether the winner's virtual valuation is no lower in the FPA than in the SPA with probability one, but rather whether this is the case in expectation. Hence, he identifies conditions under which the inner integral in (3) is positive. However, Kirkegaard (2012a) explicitly makes the point that his method may fail if there is more than one strong bidder.

4.2 Ranking auctions with large reserve prices

The possibility of bid-separation is a distinguishing feature of auctions with more than one strong bidder. Bid-separation limits how often any weak bidders wins. Even if his type is $\overline{v}_w$, he wins only if all strong bidders have types below $\widehat{v}$. Since $\widehat{v}$ depends on the reserve price, the idea is to use the latter as a lever to determine how often weak bidders win. As a result, the winner's virtual valuation in the FPA will be shown to be no smaller than in the SPA for at least some reserve prices. In other words, there are $(r, m_s, m_w) \in \mathcal{P}$ with $m_s \geq 2$. This is precisely the simple proof strategy that Maskin and Riley (2000) note cannot work in two-bidder auctions. Thus, contrary to what common intuition may suggest, auctions with several bidders may be easier to handle than auctions with just two bidders.

Recall that $\widehat{v} = k(\overline{v}_w)$ converges to $\overline{v}_w$ from above as r converges to $\overline{v}_w$ from below. In other words, for any $v \in (r, \overline{v}_w]$, $k(v|r, m_s, m_w)$ can be made arbitrarily close to v by gradually increasing r. At the same time, (4) implies that $J_w(v) > J_s(x)$ when v and x are close to $\overline{v}_w$. Thus, as r increases towards $\overline{v}_w$, $(r, m_s, m_w) \in \mathcal{P}$. Hence, the FPA outperforms the SPA when r is close enough to $\overline{v}_w$.

Intuitively, when r is close to $\overline{v}_w$ any weak bidder that bids in the auction is forced to pay close to his valuation in either auction. Hence, there is little room left to increase his bid in the FPA. Thus, he is only capable of outbidding strong bidders with slightly higher types. However, such types must have much lower virtual valuations. In other words, if a weak bidder outbids a strong bidder in such circumstances then the weak bidder is guaranteed to have a higher virtual valuation than the strong bidder. Thus, the FPA discriminates in favor of the weak bidders, but not too much.

Proposition 4 *Given $m_s \geq 2$ and $m_w \geq 1$, there exists an $\widehat{r} \in [\underline{v}_s, \overline{v}_w)$ such that*

$\Delta(r, m_s, m_w) > 0$ *for all* $r \in [\widehat{r}, \overline{v}_w)$.

Proposition 4 is a "local" result that requires minimal assumptions; it has been assumed only that $\overline{v}_s > \overline{v}_w$ and that reverse hazard rate dominance applies. Indeed, it would be sufficient to assume that reverse hazard rate dominance applies "locally" around $\overline{v}_w$, or, by continuity, that $\frac{f_s(\overline{v}_w)}{F_s(\overline{v}_w)} > f_w(\overline{v}_w)$. Note that not even first order stochastic dominance is required to hold.

Since it is a local result, Proposition 4 is silent on how large the range of reserve prices is for which the FPA can be said to dominate the SPA. Example 2 in Section 5 provides an example in which this range can be characterized. However, the main use of Proposition 4 lies in the fact that it presents just enough of a wedge to prove in Section 5 that there are configurations of type distributions in which the profit ranking flips with changes in r because in such cases the SPA is more profitable when r is small. Hence, Proposition 4 is most definitely not a global result without further restrictions on type distributions.

Appendix B extends Proposition 4 to the case with a single strong bidder, $m_s = 1$. Since bid-separation never arises in that case, arguments that are closer in spirit to Kirkegaard (2012a) must be used. Appendix B also describes conditions that are weaker than $(r, m_s, m_w) \in \mathcal{P}$, but which are still sufficient for $\Delta(r, m_s, m_w) > 0$. Finally, note also that Proposition 4 must hold if there are more than two groups of bidders with different maximum types. After all, the analysis applies when the reserve price is so high that it effectively excludes all but two groups.

4.3 Ranking auctions with many bidders

Propositions 1 and 2 imply that (r, m_s, m_w) is more likely to belong to $\mathcal{P}$ the higher r, m_s, or m_w are. The reason is that $k(v)$ decreases, meaning that the strict inequality in the definition of $\mathcal{P}$ must hold for fewer values of x (and fewer values of v if r increases too). Intuitively, the auction becomes closer to symmetric as r, m_s, or m_w increases. This means that it is less likely that a weak bidder outbids a strong bidder with a higher virtual valuation in the FPA. Thus, when a weak bidder wins, it is because he has the highest virtual valuation.

Proposition 5 *Assume* $\overline{v}_w > r' \geq r > \underline{v}_s$, $m_s' \geq m_s \geq 2$, *and* $m_w' \geq m_w \geq 1$. *Then,* $(r', m_s', m_w') \in \mathcal{P}$ *if* $(r, m_s, m_w) \in \mathcal{P}$.

Proposition 5 implies that as the number of bidders increases, the set of reserve prices for which the FPA can be proven to be preferable to the SPA weakly expands. A stronger version can be obtained under the additional assumption that F_s strictly dominates F_w in terms of the hazard rate, $F_w <_{hr} F_s$, or

$$\frac{f_s(v)}{1-F_s(v)} < \frac{f_w(v)}{1-F_w(v)} \text{ for all } v \in (\underline{v}_s, \overline{v}_w).$$

Note that $J_w(v) > J_s(v)$ for all $v \in (\underline{v}_s, \overline{v}_w]$. Thus, it follows that $(r, m_s, m_w) \in \mathcal{P}$ if $k(v)$ is sufficiently close to v for all $v \in (r, \overline{v}_w]$. Invoking Propositions 2 and 3 then imply that for any $r \in (\underline{v}_s, \overline{v}_w)$, the FPA is strictly more profitable than the SPA when sufficiently many bidders are participating in the auction.

Proposition 6 *Assume $F_w <_{hr} F_s$. Then, for any $r \in (\underline{v}_s, \overline{v}_w)$, $(r, m_s, m_w) \in \mathcal{P}$ when m_s and/or m_w is sufficiently large.*

Hazard rate dominance is used only in Proposition 6 and one later result (Proposition 8). Both deal with changes in the number of bidders, something that has gone largely unexplored in the existing literature

5 Reversals of the profit ranking

Compared to much other work on ranking asymmetric auctions, the structure imposed here is rather sparse. The configuration of type distributions has so far been endowed only with the following properties:

(i) Different maximal types; $\overline{v}_w < \overline{v}_s$.

(ii) Reverse hazard rate dominance; $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$ for all $v \in (\underline{v}_s, \overline{v}_w]$.

Proposition 6 additionally assumes:

(iii) Strict hazard rate dominance; $\frac{f_s(v)}{1-F_s(v)} < \frac{f_w(v)}{1-F_w(v)}$ for all $v \in (\underline{v}_s, \overline{v}_w)$.

In Kirkegaard (2012a) and two of Maskin and Riley's (2000) configurations, (i) and (ii) are imposed together with a stronger version of (iii). As explained in Kirkegaard (2012a), their assumptions are strong enough to guarantee that the FPA outperforms the SPA for all reserve prices in the two-bidder case and indeed whenever $m_w \geq 1$ as long as $m_s = 1$. Maskin and Riley (2000) describe a third configuration in which the SPA outperforms the FPA under the assumption that there are two bidders and no reserve price. However, their logic extends to any reserve price and any number of bidders. In that example, the inequalities in (i) and (iii) are replaced by equalities. In summary, all these configurations are characterized by profit rankings that are robust to changes in parameters. The following two examples illustrate that there are configurations for which the ranking is fragile. The first example is constructive and explains how to generate configurations in which the profit ranking may flip. The second example provides a more concrete illustration.

EXAMPLE 1 (RANKING REVERSALS): As a preliminary thought experiment, assume that $F_s(v)$ is obtained by truncating F_w on the left, such that

$$F_s(v) = \frac{F_w(v) - F_w(\underline{v}_s)}{1 - F_w(\underline{v}_s)},\ v \in [\underline{v}_s, \overline{v}_w]$$

for some truncation point $\underline{v}_s \in (\underline{v}_w, \overline{v}_w)$. It is easy to see that $\frac{F_s(v)}{F_w(v)}$ is strictly increasing on $v \in (\underline{v}_s, \overline{v}_w]$. That is, reverse hazard rate dominance applies. However, contrary to the main model, $\overline{v}_w = \overline{v}_s$. It also holds that $J_w(v) = J_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$. Finally, assume that $J_w(v)$ and $J_s(v)$ are strictly increasing in v. Then, the efficient SPA allocates the good optimally whenever it is sold. In the FPA, bid-separation does not arise in equilibrium since $\overline{v}_w = \overline{v}_s$. However, due to reverse hazard rate dominance it must hold that $k(v) > v$ for all $v \in (r, \overline{v}_w)$, for any $r \in [\underline{v}_s, \overline{v}_w)$. Thus, weak bidders win more often than is efficient. Hence, the SPA strictly outperforms the FPA. Maskin and Riley's (2000) example that demonstrates the SPA may be more profitable than the FPA is based on the same logic.

Now perturb the model. Specifically, "stretch" F_s from the support $[\underline{v}_s, \overline{v}_w]$ to the support $[\underline{v}_s, \overline{v}_w + \varepsilon]$, where $\varepsilon > 0$ is small. The new, perturbed, distribution H_s satisfies $H_s(v) = \lambda F_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$ for some $\lambda > 0$ that is strictly smaller than one but very close to one. The reverse hazard rate is unaffected on $[\underline{v}_s, \overline{v}_w]$ and so it still holds that H_s strictly reverse hazard rate dominates F_w. However, it is now the case that $\overline{v}_w < \overline{v}_w + \varepsilon = \overline{v}_s$ and $J_w(v) > J_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$. Hence, the perturbed model satisfies all the assumptions required for the previous analysis. Fix some $r \in [\underline{v}_s, \overline{v}_w)$ and some $m_s \geq 2$, $m_w \geq 1$. By continuity (see Lebrun (2002)), if ε is close enough to zero and λ close enough to one then it must still hold that the SPA outperforms the FPA. Now increase the reserve price. As the reserve price approaches $\overline{v}_w$, Proposition 4 comes into effect and the ranking thus flips.

In conclusion, the SPA outperforms the FPA when the reserve price is low enough, whereas the FPA outperforms the SPA when the reserve price is high enough. Similarly, holding r fixed, the SPA outperforms the FPA with the original set of bidders. However, by Proposition 6, the FPA will eventually come to dominate the SPA as the number of bidders increases. I am aware of no other work that has demonstrated either of these ranking reversal properties before. ▲

Example 2 (Uniform Distributions with Two Bidders): Assume now that F_w and F_s are uniform distributions and that $m_s = m_w = 1$. As mentioned, Appendix B extends Proposition 4 to the case where $m_s = 1$. The advantage of the assumption that $m_s = m_w = 1$ is that it makes it possible to invoke Kaplan and Zamir's (2012) analytical characterization of inverse bidding strategies, which in turn makes it possible to evaluate expected revenue for any r. To continue, assume that $\underline{v}_w = 0$ and $\overline{v}_w = 100$ but let $\underline{v}_s \in (0, 100)$ and $\overline{v}_s > 100$ vary. Figure 2(a) summarizes the conclusions from the resulting revenue ranking exercise. At any point on or below the curve, the SPA is more profitable than the FPA for a subset of reserve prices. Note that this occurs when $\overline{v}_s$ is sufficiently small, which is consistent with the conclusion in Example 1 for $m_s \geq 2$. In all cases, the higher the reserve price, the less likely it is that the SPA outperforms the FPA. This is consistent with Proposition 4 which implies that the FPA must outperform the SPA when the reserve price is high enough. Figure 2(b) depicts the percentage gain in moving from the SPA to the FPA as a function of r when $\underline{v}_s = 40$ and $\overline{v}_s = 105$.

The revenue differences in Figure 2(b) are small. The reason is that neither distribution have any curvature and that $\overline{v}_s$ is close to $\overline{v}_w$. Hence, the two bidders are in a sense not "too asymmetric." In Figure 2(a), the SPA is never more than 1.36% better than the FPA. As $\overline{v}_s$ increases, however, the FPA becomes increasingly more profitable relative to the SPA. For instance, if $\overline{v}_s = 150$ and $r = \underline{v}_s = 30$, then the FPA yields 5.52% higher profit than the SPA. If $\overline{v}_s = 200$, this number increases to 12.19%. These magnitudes are not unusual in the literature. Li and Riley (2007) solve for bidding strategies numerically and presents a number of examples with six asymmetric bidders. In their examples, when the SPA outperforms the FPA is does so by a small margin. On the other hand, when the FPA outperforms the SPA, there is more variability in how much better the FPA is. However, an improvement of

5-10% is not unusual, even with six bidders.

In the setting in Figure 2(b), expected revenue in the FPA ranges from 52.06 at $r = \underline{v}_s = 40$ up to a maximum of 53.37 at $r = 52.66$ and down to 7.69 at $r = \overline{v}_w = 100$. Hence, setting the reserve price too high can be very costly, whereas a reserve price that is too low is less damaging to expected revenue. Reserve prices are endogenized in Section 7. ▲

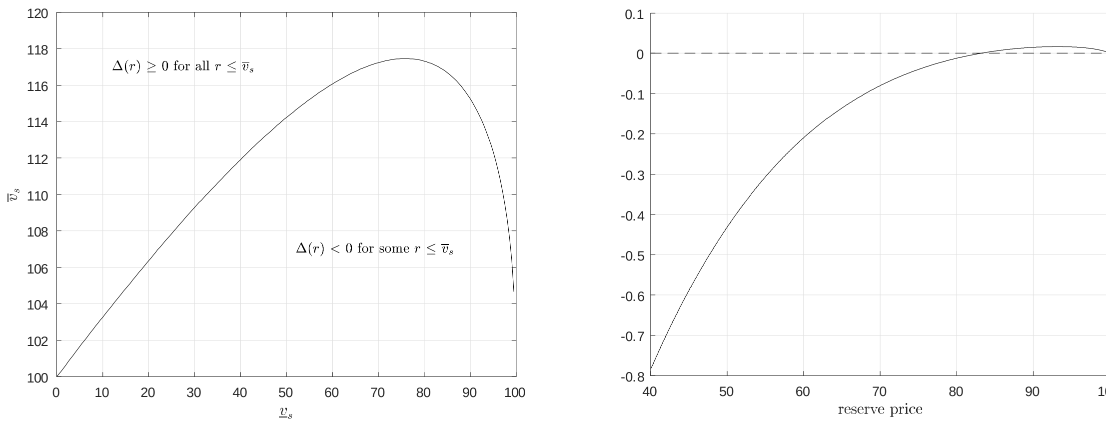

Figure 2: (a) The left panel summarizes ranking results for varying $(\underline{v}_s, \overline{v}_s)$; (b) The right panel shows the percentage gain in using the FPA when $\underline{v}_s = 40$ and $\overline{v}_s = 105$.

6 The role of the type distributions

As an interlude before turning to endogenous reserve prices, it is useful to develop a way to differentiate between various configurations of type distributions. This section provides a way of describing or parameterizing the degree of asymmetry between the two groups of bidders. Recall that $\overline{v}_s \neq \overline{v}_w$ is instrumental to the proof strategy. Holding $F_w(v)$ and $\overline{v}_w$ fixed, the idea is to use $\overline{v}_s$ to parameterize the strong bidders' distribution.

Consider some twice continuously differentiable function, $G(v)$, defined for all $v \geq \underline{v}_s$. Assume that $G(\underline{v}_s) = 0$ and that the derivative, $g(v)$, is strictly positive for any $v > \underline{v}_s$. For any $\overline{v}_s \geq \overline{v}_w$, let

$$F_s(v|\overline{v}_s) = \frac{G(v)}{G(\overline{v}_s)}, \text{ for all } v \in [\underline{v}_s, \overline{v}_s], \tag{5}$$

and note that

$$\frac{f_s(v|\overline{v}_s)}{F_s(v|\overline{v}_s)} = \frac{g(v)}{G(v)}, \text{ for all } v \in [\underline{v}_s, \overline{v}_s],$$

is independent of $\overline{v}_s$. Finally, assume that

$$\frac{g(v)}{G(v)} \geq \frac{f_w(v)}{F_w(v)}, \text{ for all } v \in (\underline{v}_s, \overline{v}_w]. \tag{6}$$

Hence, F_s dominates F_w in terms of the reverse hazard rate for any $\overline{v}_s > \overline{v}_w$. Note that the two distributions may coincide in the limit where $\overline{v}_s \to \overline{v}_w$. Moreover, this formulation is without loss of generality. That is, for any fixed F_w and $\overline{v}_s$, any F_s that satisfies the assumptions in Section 2 can be written as (5) and must satisfy (6).

Adapting Maskin and Riley's (2000) terminology, increases in $\overline{v}_s$ amounts to "stretching" the distribution. This is precisely the kind of change that occurred in Example 1. Note that the strong groups' virtual valuation,

$$J_s(v|\overline{v}_s) = v - \frac{1 - F_s(v|\overline{v}_s)}{f_s(v|\overline{v}_s)} = v - \frac{G(\overline{v}_s) - G(v)}{g(v)}$$

is strictly decreasing in $\overline{v}_s$. Hence, it becomes more profitable to discriminate against strong bidders in favor of weak bidders when $\overline{v}_s$ increases. This already suggests that the FPA is more likely to be more profitable than the SPA as $\overline{v}_s$ grows.

Next, assume that $\widehat{v} < \overline{v}_s$ for some fixed $\overline{v}_s$ value. Thus, bid-separation occurs. Stretching F_s entails adding more high types to the strong bidders' type space. It is not surprising that these new types will also separate away from weak bidders by bidding above $\overline{b}_w$. In fact, bidding behavior for existing types do not change.[11] That is, $k(v|r, m_s, m_w)$ is unchanged as $\overline{v}_s$ increases. Hence, the allocation does not change for a given type profile. Moreover, the difference in profit between the FPA and SPA can now be written as

$$\Delta(r, m_s, m_w|\overline{v}_s) = \frac{1}{G(\overline{v}_s)^{m_s}} \int_r^{\overline{v}_w} \left(\int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x|\overline{v}_s))\, dG(x)^{m_s} \right) dF_w(v)^{m_w}.$$

The factor before the integral is irrelevant for the sign of $\Delta(r, m_s, m_w|\overline{v}_s)$. The terms under the integral increase with $\overline{v}_s$. Hence, if $\Delta(r, m_s, m_w|\overline{v}_s)$ is positive, then it

[11] Technically, the reason is that the reverse hazard rate is unchanged when F_s is stretched. The system of differential equations in the common bid range is then unchanged as well.

remains positive as $\overline{v}_s$ increases. This implies that as the degree of asymmetry increases, the FPA can be proven to be superior to the SPA for more and more parameters (r, m_s, m_w).[12] This is consistent with Figure 2(a) in Example 2, where the FPA unambiguously outperforms the SPA when $\overline{v}_s$ is large enough.

Proposition 7 *Fix* $r \in (\underline{v}_s, \overline{v}_w)$. *If* $\widehat{v} < \overline{v}_s$ *and* $\Delta(r, m_s, m_w|\overline{v}_s) > 0$ *then these properties also hold as* $\overline{v}_s$ *increases.*

7 Endogenous reserve prices

The reserve price is now endogenized. It is assumed that the seller chooses r with the objective of maximizing expected profit. The exogenous parameters of the model are now z, m_s, and m_w. Let $r^{SPA}(z, m_s, m_w)$ denote the optimal reserve price in the SPA for a seller with own-use value z. If the optimal reserve price is not unique, then $r^{SPA}(z, m_s, m_w)$ denotes the *smallest* optimal reserve price. Let $r^{FPA}(z, m_s, m_w)$ denote *any* optimal reserve price in the FPA. With some abuse of notation, I write $r^{SPA}(z)$ and $r^{FPA}(z)$ whenever the number of bidders is understood to be fixed.

In either auction, the seller is pursuing one of two strategies. Specifically, she is either attempting to profit from both groups of bidders by accommodating weak bidders with a reserve price below $\overline{v}_w$, or she is focusing on extracting as much rent as possible from strong bidders by using a reserve price that is prohibitive for weak bidders. In the latter case, the two auctions are revenue equivalent since only one group of bidders is involved. This is optimal when the strong group is much stronger than the weak group, or, roughly speaking, when the asymmetry is sufficiently large. Thus, this section concentrates on settings where the asymmetry is in some sense not too large.

The first subsection considers the role of the number of bidders, while the second examines the role of z. The third subsection proves that there are configurations of type distributions where the profitability ranking flips as z changes. The companion paper, Kirkegaard (2020), complements this paper by establishing that the optimal reserve price in the FPA is sometimes below the optimal reserve price in the SPA. As a consequence, the FPA is more likely to realize gains from trade. With endogenous reserve prices, the FPA may therefore ultimately be more efficient than the SPA.

[12] On the other hand, reserve prices above $\overline{v}_w$ become more profitable too. This is one of the reasons that endogenizing the reserve price is more challenging. See Section 7.

7.1 Ranking auctions with many bidders

Recall that $J_w(\overline{v}_w) = \overline{v}_w > 0$. Assume in this subsection that $J_s(v) \geq 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$. One interpretation of the assumption is that the asymmetry between bidders is not too large. Assume moreover that z is small enough that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. These assumptions are easily verified to imply that $\Pi^{SPA}(z, r, m_s, m_w)$ and $\Pi^{FPA}(z, r, m_s, m_w)$ are non-increasing in r for $r \geq \overline{v}_w$ (recall that the auctions are equally profitable for $r \geq \overline{v}_w$). In fact, the optimal reserve price in either auction is strictly below $\overline{v}_w$.

Assume moreover that $F_w <_{hr} F_s$, so that Proposition 6 can be invoked. Thus, the FPA is strictly better than the SPA for all non-prohibitive reserve prices when there are enough bidders present. Hence, the FPA must also be strictly better than the SPA when the reserve price is endogenous and allowed to vary with the auction format.

Proposition 8 *Assume $F_w <_{hr} F_s$ and that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. Then,*

$$\Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w)$$

when m_s and/or m_w is sufficiently large. Thus, the SPA is not weakly more profitable than the FPA for all (m_s, m_w).

Proof. See Appendix A. ■

The logic behind Proposition 8 is that the FPA is "almost" efficient when there are many bidders and so $(r, m_s, m_w) \in \mathcal{P}$ for any r that is a candidate for an optimal reserve price in the SPA, given $F_w <_{hr} F_s$. However, since the FPA is "almost" efficient, it also produces "almost" the same allocations as the SPA. Hence, the two auctions are unlikely to differ much in terms of expected profit when there are many bidders at auction.

In summary, the message is not that the FPA always outperforms the SPA, although that is sometimes the case. Instead, the message is that the SPA *cannot* always dominate the FPA; any claim to the contrary is more fragile.

7.2 The role of own-use valuations

Proposition 8 implies that there is a whole range of own-use valuations for which the FPA strictly outperforms the SPA for some large enough (m_s, m_w). A partial converse is pursued in this subsection. Thus, the aim is to examine whether for *any* $(m_s, m_w) \geq (2, 1)$, there exists *some* parameter value of z such that the FPA strictly outperforms the SPA with endogenous reserve prices. Stated differently, a counterpart to Proposition 4 that now allows for endogenous reserve prices is sought.

Unfortunately, a non-trivial issue arises when the reserve price is endogenized. First, to ensure that $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$ it is necessary that $r^{SPA}(z)$ is "high enough", which generally requires z to be large. However, if z is too large, then it is no longer the case that $r^{SPA}(z) < \overline{v}_w$. Intuitively, $r^{SPA}(z)$ is typically increasing in z because higher z implies that the seller is happier to retain the object. The problem, however, is that as z increases, r^{SPA} may discontinuously jump from some value strictly below $\overline{v}_w$ to some value strictly above $\overline{v}_w$. The reason is again that the seller jumps from accommodating both groups of bidders to just concentrating on extracting as much rent as possible from the strong bidders. As z increases, the seller switches from the former to the latter approach. Consequently, there are reserve prices close to $\overline{v}_w$ that can never be rationalized in a SPA, regardless of z. Thus, it is hard in general to establish the existence of a z for which $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$.

To overcome this technical difficulty I return to the formulation of the model presented in Section 6. Starting from $\overline{v}_s = \overline{v}_w$, it is then possible to consider configurations of type distributions with "small asymmetries", or, more formally, configurations in which $\overline{v}_s$ is marginally above $\overline{v}_w$.[13] Recall that Example 1 fits this model. In this setting, it can be proven that there are own-use valuations for which the FPA strictly outperforms the SPA with endogenous reserve prices.

Proposition 9 *Assume that $F_s(\cdot|\overline{v}_s)$ and $F_w(\cdot)$ satisfy (5)–(6). Then, there is some $\overline{v}_s' > \overline{v}_w$ such that for any $\overline{v}_s \in (\overline{v}_w, \overline{v}_s')$ there exists an own-use valuation z for which*

$$\Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w)$$

for all $m_s \geq 2$, $m_w \geq 1$.

Proof. See Appendix A. ■

Example 4 in the next subsection quantifies the range of z values for which the FPA outperforms the SPA.

[13] Note that a "small asymmetry" refers only to a small distance between $\overline{v}_s$ and $\overline{v}_w$. It is possible that $F_s(v)$ and $F_w(v)$ are "far apart" on $(\underline{v}_s, \overline{v}_w]$.

7.3 Reversals of the profitability ranking

Assuming reserve prices are the same for both auction formats, Examples 1 and 2 demonstrated that the profit ranking may flip with a change in the exogenous reserve price. However, when it is endogenous, the optimal reserve price generally differs across auctions. The next examples strengthens the conclusion of Examples 1 and 2 by showing that the profit ranking, even when allowing for endogenous reserve prices, may also flip as the seller's own-use valuation changes.

Example 3 (Sensitivity to the seller's own-use valuation): Return to the set-up at the beginning of Example 1 where F_s is obtained by truncating F_w on the left, such that $\overline{v}_s$ initially coincides with $\overline{v}_w$. As mentioned, it then holds that $J_w(v) = J_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$. Assume that $J_w(\underline{v}_s) \geq 0$ and let $z = J_w(\underline{v}_s)$ to begin. Finally, assume that virtual valuations are strictly increasing. It is easy to see that the optimal reserve price in the SPA is $r^{SPA}(z) = \underline{v}_s$ and that the SPA implements the optimal auction. Thus,

$$\Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w) > \Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w).$$

By continuity, a small perturbation of F_s, obtained by marginally increasing $\overline{v}_s$, cannot change this ranking when z is held fixed at $z = J_w(\underline{v}_s)$. However, Proposition 9 proves that there must be some other z for which the FPA is strictly more profitable than the SPA with endogenous reserve prices. ▲

Example 4 (Uniform Distributions with Two Bidders): Consider the setting in Example 2, but assume now for concreteness that $\underline{v}_s = 40$. In the following, it is $\overline{v}_s$ and z that are varied. Thus, the auction problem is solved for any $(z, \overline{v}_s)$ combination. Figure 3(a) summarizes the findings. The SPA is more profitable than the FPA when $\overline{v}_s$ and z are small. However, the FPA is more profitable when $\overline{v}_s$ is large or z is in an intermediate range. Finally, if z is very large then it is optimal to set a reserve price above $\overline{v}_w$, in which case the two auctions are revenue equivalent.

In either auction, a small reserve price is optimal when z is small. Although the optimal reserve price is slightly different in the SPA and the FPA, this is not enough to overturn the conclusion from Figure 2(a) that the SPA is more profitable than the FPA when $\overline{v}_s$ is small. However, as z increases, small reserve prices are no longer optimal in either auction. This means that the SPA cannot outperform the FPA.

The FPA is thus strictly better for a range of own-use valuations. Eventually, as z becomes large enough, it is better to exclude the weak bidder and the SPA and FPA are then equally profitable. ▲

Examples 3 and 4 establishes that the seller's own-use valuation may be crucial even when selecting among simple auction formats like the SPA and the FPA. This fact represents a challenge to the applied literature where the seller's own-use valuation need not be known. It can perhaps be argued that z can be inferred from the observed reserve price that the seller is using in the real world. Even given this optimistic premise, however, it may be impossible for the econometrician to determine whether the auction currently in use should be replaced with the alternative auction format. The reason is that the optimal reserve price in the counterfactual auction may be below that used in the real auction. The problem is that the bid data is necessarily truncated by the current reserve price. Thus, the econometrician does not necessarily have access to data that would allow him to calculate optimal profit in the counterfactual auction.

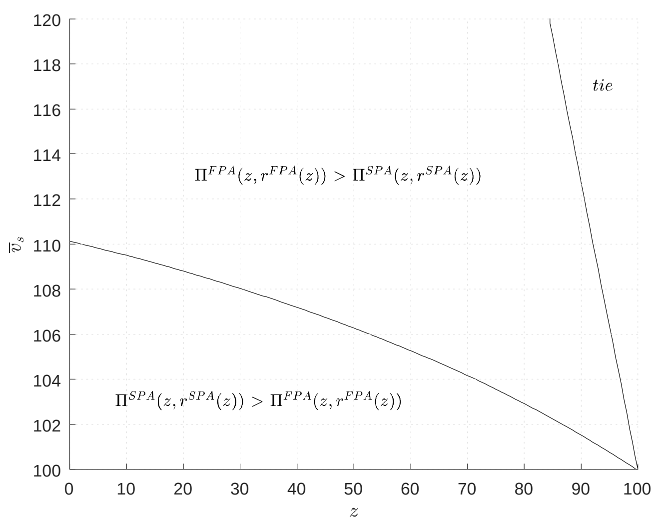

Figure 3: Optimal auction choice with endogenous reserve prices.

8 Conclusion

Progress on ranking the profit of different auctions in the presence of asymmetries has been slow. This paper cautions that a robust ranking cannot be obtained for all configurations of type distributions. Parameters such as reserve prices, the number of bidders, and the seller's own-use valuation may influence the ranking. By extension, the paper puts a renewed emphasis on the seller's own-use valuation. It may play a more central role in selecting the best auction format than suggested by existing theory.

These results suggest some caution is prudent when interpreting various findings in the empirical literature. When conducting the counterfactual analysis described in the introduction, it is rare that changes in the reserve price are examined as well. Since the profit ranking may be sensitive to the reserve price, it may be worthwhile to augment counterfactual studies with a robustness check along this dimension of auction design. More problematically, the best design may depend on the seller's own-use valuation, which is less likely to be known.

The model assumes participation is exogenous, yet it is not without implications for the issue of entry. The value of attracting more participation is well-recognized; see e.g. Bulow and Klemperer (1996). As the profit ranking may also depend on the composition of bidders, any steps taken to encourage entry should at the very least be accompanied by an examination of whether a change in auction design at the same time is called for.

The basic intuition also carries over to auctions with multiple identical units in which each bidder demands only a single unit. Then, bid-separation may arise in the discriminatory auction whenever there are more strong bidders than units. Therefore, the logic that led to Proposition 4 still applies. Thus, there are reserve prices for which the discriminatory auction is more profitable than any efficient auction.

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Appendix A: Omitted proofs

Describing the problem: For completeness, I first outline both formulations of the problem. To begin, let $\varphi_i(b)$ denote bidder i's inverse bidding strategy, $b \in [r, \bar{b}_i]$, $i = s, w$. On the range of bids where both groups of bidders are active, $[r, \bar{b}_w]$, $\varphi_w(b)$ and $\varphi_s(b)$ solve the system of differential equations described by

$$\frac{d}{db} \ln F_i(\varphi_i(b)) = \frac{1}{m_s + m_w - 1} \left[\frac{m_j}{\varphi_j(b) - b} - \frac{m_j - 1}{\varphi_i(b) - b} \right], \tag{7}$$

$i, j = s, w$, $i \neq j$, with boundary conditions $\varphi_w(\bar{b}_w) = \bar{v}_w$ and $\varphi_s(\bar{b}_w) = \hat{v}$. Note that if $\hat{v} < \bar{v}_s$, then $\varphi_w'(\bar{b}_w) = 0$, by (2). Lebrun (2006) proves that $\varphi_i'(b) > 0$ for all interior bids, however.

Second, consider the formulation of the problem in terms $b_w(v)$ and $k(v)$. If his type is v, a weak bidder's problem can be thought of as deciding which type, x, to mimic. His problem is thus to maximize

$$(v - b_w(x)) F_s(k(x))^{m_s} F_w(x)^{m_w - 1}.$$

Similarly, a strong bidder with type $k(v)$ who bids in the common range maximizes

$$(k(v) - b_w(x))\, F_s(k(x))^{m_s - 1} F_w(x)^{m_w}.$$

By definition of equilibrium, bidders' payoffs are maximized when $x = v$. When $v \in (r, \bar{v}_w)$, the first order conditions yield the system of differential equations

$$\begin{aligned} k'(v) &= \frac{F_s(k(v))}{f_s(k(v))} \frac{f_w(v)}{F_w(v)} T(k(v), b_w(v), v) \\ b_w'(v) &= \frac{f_w(v)}{F_w(v)} (k(v) - b_w(v)) \left[(m_s - 1)\, T(k(v), b_w(v), v) + m_w \right], \end{aligned} \tag{8}$$

where

$$T(k, b_w, v) = \frac{m_w \frac{k - b_w}{v - b_w} - (m_w - 1)}{m_s - (m_s - 1) \frac{k - b_w}{v - b_w}}$$

To compare this formulation of the problem with the previous one, the boundary conditions are that $k(\bar{v}_w) = \hat{v}$ and $b_w(\bar{v}_w) = \bar{b}_w$.[14] Note that $T(k, b_w, v) \gtreqless 1$ if and

[14] In equilibrium, $k'(v) > 0$ and $b_w'(v) > 0$. Note, however, that if $\hat{v} < \bar{v}_s$ then $T(k(v), b_w(v), v)$

only if $k \gtreqless v$. Likewise, holding b_w and v fixed, $T(k, b_w, v)$ is strictly increasing in k. It also holds that $\frac{\partial T(k,b_w,v)}{\partial b_w} \gtreqless 0$ if and only if $k \gtreqless v$. These properties will be used repeatedly. ▲

Proof of Lemma 1. Given these preliminaries it is now possible to prove Lemma 1. Recall that $k(\overline{v}_w) > \overline{v}_w$. To illustrate the proof idea, assume first that the inequality in (1) is strict. Assume there exists some $v_0 \in (r, \overline{v}_w]$ for which $k(v_0) = v_0$. Since $T = 1$ at such a point,

$$k'(v_0) = \frac{F_s(v_0)}{f_s(v_0)} \frac{f_w(v_0)}{F_w(v_0)} < 1.$$

Thus, increasing v beyond v_0 leads to the conclusion that $k(v) \leq v$. However, this contradicts the equilibrium feature that $k(\overline{v}_w) > \overline{v}_w$. The idea is the same when the inequality in (1) is weak. More formally, assume once again that there exists some $v_0 \in (r, \overline{v}_w)$ for which $k(v_0) = v_0$. Based on this "initial condition", the next step is to obtain the solution to the system of differential equations as v increases beyond v_0 (the solution to this initial value problem is unique given the differentiability assumptions imposed on the primitives). To begin, the guess is made that the solution satisfies $k(v) \leq v$ for all $v \geq v_0$. Then, $T \leq 1$, and it follows that

$$\frac{d}{dv} \ln F_s(k(v)) = \frac{f_s(k(v))}{F_s(k(v))} k'(v) \leq \frac{f_w(v)}{F_w(v)} = \frac{d}{dv} \ln F_w(v),$$

or

$$\frac{d}{dv} \ln \frac{F_s(k(v))}{F_w(v)} \leq 0$$

independently of $b_w(v)$. By Gronwall's inequality, the actual solution is then bounded above by the solution that would be obtained if the above inequality had been replaced by an equality, in which case $\ln \frac{F_s(k(v))}{F_w(v)}$ would be constant. Hence, using the initial condition that $k(v_0) = v_0$,

$$\ln \frac{F_s(k(v))}{F_w(v)} \leq \ln \frac{F_s(v_0)}{F_w(v_0)}. \tag{9}$$

However, since $v \geq v_0$ reverse hazard rate dominance implies that

$$\frac{F_s(v)}{F_w(v)} \geq \frac{F_s(v_0)}{F_w(v_0)},$$

goes to infinity as v approaches $\overline{v}_s$, by (2).

and so (9) necessitates that $k(v) \leq v$. Thus, the initial guess that $k(v) \leq v$ for all $v \geq v_0$ when $k(v_0) = v_0$ is verified. The proof is then completed in the same manner as before. In particular, the implication that $k(\overline{v}_w) \leq \overline{v}_w$ violates the equilibrium property that $k(\overline{v}_w) > \overline{v}_w$. Hence, there can be no $v_0 \in (r, \overline{v}_w)$ for which $k(v_0) = v_0$. By continuity, it then follows that $k(v) > v$ for all $v \in (r, \overline{v}_w]$. ■

Proof of Proposition 1. I first establish that the initial conditions are that $b_w(r) = r$ and $k(r) = r$. Lebrun (2006) shows that in general $\varphi_i(r) = r$ for all but at most one bidder i; see his conditions ($2'$) and ($2''$) along with his discussion on page 143. Stated differently, it is possible that $\varphi_i(r) > r$ for exactly one bidder, such that bidder i has a mass of types that bids r. However, since strategies within any given group is symmetric and $m_s \geq 2$, no strong bidder can bid r for a mass of types. The same holds for weak bidders if $m_w \geq 2$. This leaves the case where $m_w = 1$. Compared to Lebrun (2006), however, here it is assumed that reverse hazard rate dominance applies. By Lemma 1, the weak bidder is more aggressive than the strong bidders, for comparable types. Thus, the weak bidder cannot, in equilibrium, be bidding r for a mass of types. In short, it must hold that $\varphi_i(r) = r$ for all bidders in the current model. Equivalently, the initial conditions to the system in (8) are that $k(r) = r$ and $b_w(r) = r$.

Let $\widehat{v}$ denote the strong bidders' cut-off type and $\overline{b}_w$ the weak bidders' maximum bid when the reserve price is r. Let $\widehat{v}'$ and $\overline{b}_w'$ denote their counterparts when the reserve price increases to r'. Note first that if $\overline{b}_w = \overline{b}_w'$ then $\widehat{v} = \widehat{v}'$, by (2). The system of differential equations are then characterized by the same boundary conditions regardless of whether the reserve price is r or r'. Thus, the system is the same on $b \in (r', \overline{b}_w]$ in either case. Given the differentiability assumptions imposed on the primitives, the unique solution to the two problems must then coincide on $b \in (r', \overline{b}_w]$. Hence, in the limit, $b_w(r'|r') = b_w(r'|r)$. However, the initial conditions when the reserve price is r' requires $b_w(r'|r') = r'$, whereas equilibrium bidding when the reserve price is $r < r'$ satisfies $b_w(r'|r) < r'$. This contradicts the previous conclusion that $b_w(r'|r') = b_w(r'|r)$. Thus, in equilibrium, $\overline{b}_w \neq \overline{b}_w'$.

Consider next the possibility that $\overline{b}_w > \overline{b}_w'$, implying that $\widehat{v}' \geq \widehat{v}$, by (2). Assume first that $\widehat{v}' > \widehat{v}$. Hence, for v close to $\overline{v}_w$, $k(v|r')$ is strictly above $k(v|r)$ while $b_w(v|r')$ is strictly below $b_w(v|r)$, or $k(\overline{v}_w|r') = \widehat{v}' > \widehat{v} = k(\overline{v}_w|r)$ and $b_w(\overline{v}_w|r') = \overline{b}_w' < \overline{b}_w = b_w(\overline{v}_w|r)$. Reducing v from $\overline{v}_w$, find the nearest value, v', (if one exists) where one of the new endogenous functions crosses its old counterpart. The argument in the previous paragraph rules out that $k(v'|r') = k(v'|r)$ and $b_w(v'|r') = b_w(v'|r)$ at the same time. There are two remaining cases. Assume $b_w(v'|r') = b_w(v'|r)$ but $k(v'|r') > k(v'|r)$. Then, from (8), $b'_w(v'|r') > b'_w(v'|r)$. This contradicts that $b_w(v|r') < b_w(v|r)$ for $v > v'$. Assume instead that $k(v'|r') = k(v'|r)$ but $b_w(v'|r') < b_w(v'|r)$. Then, again from (8), $k'(v'|r') < k'(v'|r)$ if $k(v'|r') = k(v'|r) > v'$. However, this contradicts that $k(v|r') > k(v|r)$ for $v > v'$.

Next, assume that $\overline{b}_w > \overline{b}'_w$ but that $\widehat{v}' = \widehat{v}$. This necessitates $\widehat{v}' = \widehat{v} = \overline{v}_s$. It can now be seen that $k(v|r)$ is steeper than $k(v|r')$ near $\overline{v}_w$. Hence, $k(v|r') > k(v|r)$ for v close to, but strictly below, $\overline{v}_w$. By continuity, it is also the case that $b_w(v|r') < b_w(v|r)$ in such a neighborhood. The previous arguments can then be repeated to obtain a contradiction.

Hence, it has now been shown that $\overline{b}_w < \overline{b}'_w$, thereby implying that $\widehat{v}' \leq \widehat{v}$. Stated differently, $b_w(\overline{v}_w|r) < b_w(\overline{v}_w|r')$ and $k(\overline{v}_w|r) \geq k(\overline{v}_w|r')$. Moreover, either $k(\overline{v}_w|r) > k(\overline{v}_w|r')$ or $k(v|r)$ is flatter than $k(v|r')$ near $\overline{v}_w$. In either case, $b_w(v|r) < b_w(v|r')$ and $k(v|r) > k(v|r')$ when v is close to $\overline{v}_w$. Arguments like those above can then be used to prove that these inequalities are unchanged as v is reduced from $\overline{v}_w$ to r'. ■

Proof of Proposition 2. Consider changes in m_w first. Let $\widehat{v}$ and $\widehat{v}'$ denote the cut-off types when the composition of bidders is (m_s, m_w) and (m_s, m'_w), respectively. Let $\overline{b}_w$ and $\overline{b}'_w$ denote weak bidders' maximum bid in the two cases. Hubbard and Kirkegaard (2019, Proposition 2) have shown that if $\widehat{v} < \overline{v}_s$, as assumed, then $\widehat{v}' < \widehat{v}$.[15] Thus, $k(\overline{v}_w|r, m_s, m'_w) < k(\overline{v}_w|r, m_s, m_w)$. Starting at $\overline{v}_w$, reduce v until the first point is reached (if one exists) where $k(v'|r, m_s, m'_w) = k(v'|r, m_s, m_w)$, with $v' > r$. Recall that by Lemma 1, $k(v'|r, m_s, m_w) > v'$. Assume first that $b_w(v'|r, m_s, m'_w) \geq b_w(v'|r, m_s, m_w)$. Then,

$$\frac{k(v'|r, m_s, m'_w) - b_w(v'|r, m_s, m'_w)}{v' - b_w(v'|r, m_s, m'_w)} \geq \frac{k(v'|r, m_s, m_w) - b_w(v'|r, m_s, m_w)}{v' - b_w(v'|r, m_s, m_w)} > 1.$$

Combined with $m'_w > m_w$ these inequalities ensure that $k'(v'|r, m_s, m'_w) > k'(v'|r, m_s, m_w)$. However, this contradicts the fact that $k(v|r, m_s, m'_w) < k(v|r, m_s, m_w)$ at $v > v'$. Assume next that $b_w(v'|r, m_s, m'_w) < b_w(v'|r, m_s, m_w)$. Since $k(v'|r, m_s, m'_w) = k(v'|r, m_s, m_w)$,

[15] The statement of Hubbard and Kirkegaard's (2019) result assumes that $m_w \geq 2$. However, as explained earlier, this assumption can be weakened to $m_w \geq 1$ once reverse hazard rate dominance is assumed.

it must also hold that

$$\begin{aligned} b_s(k(v'|r,m_s,m_w')|r,m_s,m_w') &= b_w(v'|r,m_s,m_w') \\ &< b_w(v'|r,m_s,m_w) \\ &= b_s(k(v'|r,m_s,m_w)|r,m_s,m_w), \end{aligned}$$

where b_s stands for the strong bidders' strategy. Letting $\varphi_i(b|r,m_s,m_w)$ and $\varphi_i(b|r,m_s,m_w')$ denote the inverse bidding strategy of a bidder in group i, $i = s,w$, there must now exist some b for which $\varphi_s(b|r,m_s,m_w') > \varphi_s(b|r,m_s,m_w)$ and $\varphi_w(b|r,m_s,m_w') > \varphi_w(b|r,m_s,m_w)$. However, this is impossible as established in the proof of Hubbard and Kirkegaard's (2019) Proposition 2. Hence, there can be no $v' \in (r,\overline{v}_w)$ for which $k(v'|r,m_s,m_w') = k(v'|r,m_s,m_w)$. Since $k(\overline{v}_w|r,m_s,m_w') < k(\overline{v}_w|r,m_s,m_w)$, continuity then implies that $k(v|r,m_s,m_w') < k(v|r,m_s,m_w)$ for all $v \in (r,\overline{v}_w]$. The proof of the result for changes in m_s is analogous. ■

Proof of Proposition 3. Lemma 1 establishes the lower bound that $k(v) > v$ for all $v \in (r,\overline{v}_w]$. An upper bound on $k(v)$ is derived next. The proof then concludes by showing that the upper bound converges to v as the number of bidders goes to infinity.

Using (7) and the condition that $\varphi_w'(b) \geq 0$ yield the conclusion that

$$\frac{m_s}{k(v)-b_w(v)} - \frac{m_s-1}{v-b_w(v)} \geq 0$$

or

$$k(v) \leq \frac{m_s}{m_s-1}v - \frac{1}{m_s-1}b_w(v) \tag{10}$$

for all $v \in (r,\overline{v}_w]$. Since $b_w(v)$ is bounded above by v, the last term in (10) goes to zero as $m_s \to \infty$. Since the first term converges to v, it now follows that $k(v) \to v$ as $m_s \to \infty$.

Next, consider changes in m_w instead. In equilibrium, $b_w(v) \leq v$. At the same time, it follows from Myerson (1981) that for any $v \in (r,\overline{v}_w]$,

$$(v-b_w(v))\,F_w(v)^{m_w-1}F_s(k(v))^{m_s} = \int_r^v F_w(x)^{m_w-1}F_s(k(x))^{m_s}dx$$

or

$$\begin{aligned} b_w(v) &= v - \int_r^v \left(\frac{F_w(x)}{F_w(v)}\right)^{m_w-1} \left(\frac{F_s(k(x))}{F_s(k(v))}\right)^{m_s} dx \\ &\geq v - \int_r^v \left(\frac{F_w(x)}{F_w(v)}\right)^{m_w-1} dx \to v \text{ as } m_w \to \infty. \end{aligned}$$

Thus, $b_w(v) \to v$ as $m_w \to \infty$. Once again, (10) now implies that $k(v) \to v$ as $m_w \to \infty$. ■

Proof of Proposition 8. Assume that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. Then, regardless of (m_s, m_w), the optimal reserve price in either auction is strictly below $\overline{v}_w$. Proposition 6 implies that $(r, m_s, m_w) \in \mathcal{P}$ for all $r \in (\underline{v}_s, \overline{v}_w)$ when m_s and/or m_w is sufficiently large. Hence, $\Pi^{FPA}(z, r, m_s, m_w) > \Pi^{SPA}(z, r, m_s, m_w)$ for all $r \in (\underline{v}_s, \overline{v}_w)$. Thus, if $r^{SPA}(z, m_s, m_w) > \underline{v}_s$ then the FPA is strictly more profitable, even without changing the reserve price. Similarly, if $r^{SPA}(z, m_s, m_w) = \underline{v}_s$ then it still holds, by continuity of Π^{SPA}, that $\Pi^{FPA}(z, r, m_s, m_w) > \Pi^{SPA}(z, \underline{v}_s, m_s, m_w)$ for some r close to $\underline{v}_s$. Thus, regardless of what the exact optimal reserve price is in the SPA, the FPA with an optimal reserve price is strictly more profitable. ■

Proof of Proposition 9. The proof proceeds in several steps.

STEP 1: Consider a setting in which the smallest optimal reserve price in the SPA is below $\overline{v}_w$ and assume that $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$. Then, following the same logic as in the proof of Proposition 8, the FPA must be strictly more profitable than the SPA. The reason is that the seller can guarantee herself a strictly higher expected profit in the FPA than in the SPA simply by using the same reserve price in the former as in the latter. Adjusting the reserve price in the FPA is just an additional benefit. More formally,

$$\begin{aligned} \Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w) &\geq \Pi^{FPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w) \\ &> \Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w), \end{aligned}$$

where the strict inequality follows from $(r^{SPA}(z, m_s, m_w), m_s, m_w) \in \mathcal{P}$. The proof relies on constructing a setting where the above observation can be invoked.

STEP 2: Note first that

$$\frac{\partial J_s(v|\overline{v}_s)}{\partial v} = 2 + \frac{g'(v)}{g(v)} \frac{G(\overline{v}_s) - G(v)}{g(v)}.$$

By assumption, $g(\cdot) > 0$. Likewise, since $g'(\cdot)$ is continuous by assumption, $g'(v)$ is bounded. Thus, $J_s(v|\overline{v}_s)$ is strictly increasing in v when v is close to $\overline{v}_s$. Similarly, $J_w(v)$ is strictly increasing in v when v is close to $\overline{v}_w$. By continuity, when $\overline{v}_s$ is close to $\overline{v}_w$ there thus exists some $v' < \overline{v}_w$ such that $J_w(v)$ and $J_s(v|\overline{v}_s)$ are both strictly increasing for all v between v' and $\overline{v}_w$ and $\overline{v}_s$, respectively. Next, recall that $J_s(\overline{v}_w|\overline{v}_s) < J_w(\overline{v}_w)$ whenever $\overline{v}_s > \overline{v}_w$. Thus, there also exists some $v'' < \overline{v}_w$ such that $J_s(v|\overline{v}_s) < J_w(v)$ for all $v \in (v'', \overline{v}_w]$. To clarify, both v' and v'' depend on $\overline{v}_s$. For any $\overline{v}_s$ close to $\overline{v}_w$, consider now the set of types between $\max\{v', v''\}$ and $\overline{v}_w$. For any type, v, in this set, there exists a unique $\tau > v$ that solves $J_w(v) = J_s(\tau|\overline{v}_s)$ such that virtual valuations are equated. Let $\tau(v|\overline{v}_s)$ denote the resulting function. Since $J_w(v)$ and $J_s(v|\overline{v}_s)$ are strictly increasing, $\tau(v|\overline{v}_s)$ is also strictly increasing and differentiable. In the first step, it is shown how $\tau(v|\overline{v}_s)$ and $k(v)$ can, under one specific condition, be compared by bounding the former from below and the latter from above.

Note that

$$\frac{\partial \tau(v|\overline{v}_s)}{\partial v} = J_w'(v) \left(\frac{\partial J_s(\tau|\overline{v}_s)}{\partial \tau} \right)^{-1}.$$

Recall that $\tau(\overline{v}_w|\overline{v}_s) = \overline{v}_w$ in the limit where $\overline{v}_s = \overline{v}_w$. Hence, when $\overline{v}_s = \overline{v}_w$

$$\frac{\partial \tau(v|\overline{v}_w)}{\partial v}_{|v=\overline{v}_w} = 1 < \frac{m_s}{m_s - 1}.$$

Thus, for any (m_s, m_w), there is a set of $(v, \overline{v}_s)$, with $v < \overline{v}_w < \overline{v}_s$, close to $(\overline{v}_w, \overline{v}_w)$ for which $\frac{\partial \tau(v|\overline{v}_s)}{\partial v} < \frac{m_s}{m_s - 1}$. On this set, $\tau(v|\overline{v}_s)$ is thus bounded below by

$$\underline{\tau}(v|\overline{v}_s) = \tau(\overline{v}_w|\overline{v}_s) + \frac{m_s}{m_s - 1}(v - \overline{v}_w), \tag{11}$$

where $\underline{\tau}(\overline{v}_w|\overline{v}_s) = \tau(\overline{v}_w|\overline{v}_s)$.

Together, (7) and the equilibrium property that $\varphi_w'(b) \geq 0$ imply that

$$\frac{m_s}{k(v) - b_w(v)} - \frac{m_s - 1}{v - b_w(v)} \geq 0$$

or, consistent with (2),

$$k(v) \leq \frac{m_s}{m_s - 1} v - \frac{1}{m_s - 1} b_w(v).$$

Since $b_w(v) \geq r$, $k(v)$ is bounded above by

$$\overline{k}(v) = \frac{m_s}{m_s - 1} v - \frac{1}{m_s - 1} r.$$

Now, since $\underline{\tau}(v|\overline{v}_s)$ and $\overline{k}(v)$ have the same slope, it follows that **if** $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ then $\underline{\tau}(v|\overline{v}_s) > \overline{k}(v)$ for all $v \geq r$. In this case,

$$\tau(v|\overline{v}_s) \geq \underline{\tau}(v|\overline{v}_s) > \overline{k}(v|r) \geq k(v|r)$$

and the monotonicity of $J_s(v|\overline{v}_s)$ then implies that $(r, m_s, m_w) \in \mathcal{P}$. Hence, the argument in Step 1 of the proof applies if the condition that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ is satisfied when $r = r^{SPA}(z)$. Thus, it remains to investigate $r^{SPA}(z)$, which evidently depends on z, and the condition that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$.

Step 3: The assertion in the proposition is proven by constructing a specific own-use valuation that works when $\overline{v}_s$ is close enough to $\overline{v}_w$. To this end, for any $\overline{v}_s$ close to $\overline{v}_w$, define $z(\overline{v}_s) = J_s(\overline{v}_w|\overline{v}_s) > 0$. For a fixed $\overline{v}_s$, assume the seller's own-use valuation is $z(\overline{v}_s)$. Note that $z(\overline{v}_w) = \overline{v}_w$. Thus, the optimal reserve price in either auction is exactly $\overline{v}_w$ when $\overline{v}_s = \overline{v}_w$. Note also that $z'(\overline{v}_s) < 0$. Hence, $z(\overline{v}_s) < \overline{v}_w$ whenever $\overline{v}_s > \overline{v}_w$.

Since $J_s(v|\overline{v}_s)$ is strictly increasing in v when v and $\overline{v}_s$ are close to $\overline{v}_w$, it holds that $z(\overline{v}_s) - J_s(v|\overline{v}_s) < 0$ for all $v > \overline{v}_w$. The implication is that the optimal reserve price in the SPA is strictly below $\overline{v}_w$ whenever $\overline{v}_s$ is above $\overline{v}_w$. At such reserve prices,

$$\begin{aligned}\Pi^{SPA}(z, r, m_s, m_w) &= zF_s(r)^{m_s} F_w(r)^{m_w} + m_w \int_r^{\overline{v}_w} J_w(v) F_w(v)^{m_w - 1} F_s(v)^{m_s} f_w(v) dv \\ &+ m_s \int_r^{\overline{v}_w} J_s(v) F_s(v)^{m_s - 1} F_w(v)^{m_w} f_s(v) dv + m_s \int_{\overline{v}_w}^{\overline{v}_s} J_s(v) F_s(v)^{m_s - 1} f_s(v) dv.\end{aligned}$$

Thus, any optimal reserve price in the SPA, denoted $r(\overline{v}_s)$, must satisfy the first order condition

$$m_s \left[z - J_s(r|\overline{v}_s)\right] \frac{g(r)}{G(r)} + m_w \left[z - J_w(r)\right] \frac{f_w(r)}{F_w(r)} = 0.$$

When $\overline{v}_s = \overline{v}_w$, the first order condition is satisfied at $r(\overline{v}_w) = \overline{v}_w$. By continuity, when $\overline{v}_s$ is marginally above $\overline{v}_w$, $r(\overline{v}_s)$ must remain close to $\overline{v}_w$. Thus, $J_s(v|\overline{v}_s)$ and $J_w(v)$ are strictly increasing in v for all $v \geq r$. Hence, $z(\overline{v}_s) - J_s(r|\overline{v}_s) > z(\overline{v}_s) - J_s(\overline{v}_w|\overline{v}_s) = 0$. To satisfy the first order condition it is then necessary that $z(\overline{v}_s) - J_w(r) < 0$. Consequently, $J_w(r) > z(\overline{v}_s) > J_s(r|\overline{v}_s)$ or $J_w(r) > J_s(\overline{v}_w|\overline{v}_s) > J_s(r|\overline{v}_s)$. Monotonicity then implies that $J_w(v) > J_s(v|\overline{v}_s)$ for all $v \in [r, \overline{v}_w]$. Thus, the analysis in Step 2 is valid, except that the condition that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ has not yet been verified. Thus, the last step of the proof is to prove that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ when $\overline{v}_s$ is marginally above $\overline{v}_w$.

Step 4: Given $z(\overline{v}_s) = J_s(\overline{v}_w|\overline{v}_s)$, it is straightforward to show that when $\overline{v}_s = \overline{v}_w$,

$$r'(\overline{v}_w) = -\frac{m_w}{2}\frac{1}{m_s\frac{g(\overline{v}_w)}{G(\overline{v}_w)}\frac{F_w(\overline{v}_w)}{f_w(\overline{v}_w)} + m_w} \geq -\frac{1}{2}\frac{m_w}{m_s + m_w} > -\frac{m_s - 1}{2}, \tag{12}$$

where the first inequality comes from reverse hazard rate dominance and the second inequality from the fact that $m_s \geq 2$. Given the optimal reserve price in the SPA, write the bound on $k(v)$ as

$$\overline{k}(v|r(\overline{v}_s)) = \frac{m_s}{m_s - 1}v - \frac{1}{m_s - 1}r(\overline{v}_s),$$

with

$$\frac{\partial\overline{k}(\overline{v}_w|r(\overline{v}_s))}{\partial\overline{v}_s}_{|\overline{v}_s=\overline{v}_w} = -\frac{1}{m_s - 1}r'(\overline{v}_s) < \frac{1}{2},$$

by (12).

In contrast,

$$\frac{\partial\tau(v|\overline{v}_s)}{\partial\overline{v}_s} = \frac{g(\overline{v}_s)}{g(v)}\left(\frac{\partial J_s(\tau|\overline{v}_s)}{\partial\tau}\right)^{-1}.$$

Evaluated at $\tau(\overline{v}_w|\overline{v}_s)$, the term in the parenthesis reduces to 2 when $\overline{v}_s = \overline{v}_w$. Hence, by (11)

$$\frac{\partial\underline{\tau}(\overline{v}_w|\overline{v}_s)}{\partial\overline{v}_s}_{|\overline{v}_s=\overline{v}_w} = \frac{\partial\tau(\overline{v}_w|\overline{v}_s)}{\partial\overline{v}_s}_{|\overline{v}_s=\overline{v}_w} = \frac{1}{2}.$$

Since $\underline{\tau}(\overline{v}_w|\overline{v}_w) = \overline{v}_w = \overline{k}(\overline{v}_w|r(\overline{v}_w))$, it follows that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w|r(\overline{v}_s))$ when $\overline{v}_s$ is marginally above $\overline{v}_w$. By the argument in Step 2 (which is valid by Step 3), $(r(\overline{v}_s), m_s, m_w) \in \mathcal{P}$ when $\overline{v}_s$ is marginally above $\overline{v}_w$. The proposition now follows by invoking Step 1. ■

Appendix B: Extensions

B.1 Auctions with one strong bidder

Kirkegaard's (2012a) approach accommodates any $m_w \geq 1$. However, it necessitates that $m_s = 1$. Moreover, he imposes stronger assumptions on the relationship between F_s and F_w. This subsection establishes that these additional assumptions are not required in order to extend Proposition 4 to the $m_s = 1$ case. Thus, I will assume only that *(i)* $\overline{v}_s > \overline{v}_w$, *(ii)* F_s dominates F_w in terms of the reverse hazard rate, and, for expositional simplicity, that *(iii)* $J_s(v)$ is strictly increasing.

Bid-separation never arises when there is just one strong bidder. Thus, $k(\overline{v}_w) = \widehat{v} = \overline{v}_s$ is the same regardless of the reserve price. However, it is easy to see from the system in (8) that $\overline{b}_w$ is strictly increasing in r. This in turn means that $k'(\overline{v}_w)$ becomes larger and larger as r increases. Since $\overline{b}_w > r$, it also holds that $\overline{b}_w$ converges to $\overline{v}_w$ as r converges to $\overline{v}_w$. Thus, from (8), $k'(\overline{v}_w)$ can be made arbitrarily large simply by selecting a reserve price that is sufficiently close to $\overline{v}_w$.

When the weak bidders' type, v, is sufficiently high – such that $J_w(v) > J_s(v)$ – there must exist some $\kappa > v$ for which

$$\int_v^{\kappa(v)} \left(J_w(v) - J_s(x)\right) dF_s(x) = 0.$$

The assumption that $J_s(v)$ is strictly increasing implies that $\kappa(v)$ is unique and that

$$\int_v^{k(v|r,m_s,m_w)} \left(J_w(v) - J_s(x)\right) dF_s(x) > 0$$

as long as $k(v) \in (v, \kappa(v))$. It can be verified that $\kappa(\overline{v}_w) = \overline{v}_s$. Now, $\kappa(v)$ is independent of r, whereas $k(v)$ depends on r. Moreover, $k(\overline{v}_w) = \kappa(\overline{v}_w)$. Since $k'(\overline{v}_w)$ can be made arbitrarily large by letting r converge to $\overline{v}_w$, it now follows that there exists large r for which

$$k(v|r, m_s, m_w) \in (v, \kappa(v)) \text{ for all } v \in (r, \overline{v}_w).$$

By (3), the FPA outperforms the SPA at such a reserve price. Thus, Proposition 4 has now been extended.

B.2 Weaker sufficient conditions

Assume again that $m_s \geq 2$. As emphasized earlier, the condition that $(r, m_s, m_w) \in \mathcal{P}$ is sufficient but not necessary to conclude that $\Delta(r, m_s, m_w) > 0$. For instance, the weaker condition that (r, m_s, m_w) belongs to

$$\widehat{\mathcal{P}} = \left\{ (r, m_s, m_w) \middle| \int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x))\, dF_s(x)^{m_s} > 0 \text{ for all } v \in (r, \overline{v}_w] \right\}$$

is sufficient to obtain the same ranking, as can be seen from (3). Replacing $\mathcal{P}$ by $\widehat{\mathcal{P}}$ is analogous to how Kirkegaard (2012a) refines Maskin and Riley's (2000) mechanism design argument.

Recall that $J_w(v) > J_s(x)$ when v and x are both close to $\overline{v}_w$. Thus,

$$\int_v^k (J_w(v) - J_s(x))\, dF_s(x)^{m_s} \tag{13}$$

is positive if v and k are close to $\overline{v}_w$. Consequently, when r is close to $\overline{v}_w$, $(r, m_s, m_w) \in \widehat{\mathcal{P}}$. Thus, a counterpart to Proposition 4 exists in which $\mathcal{P}$ is replaced by $\widehat{\mathcal{P}}$.

Appendix C: Excerpt from Hubbard and Kirkegaard (2019) – NOT FOR PUBLICATION

The current paper refers to Propositions 2 in Hubbard and Kirkegaard (2019). For convenience, this appendix reiterates this results along with its proof, translated into the notation of the current paper.

Remark: Reverse hazard rate dominance is not assumed in Hubbard and Kirkegaard. Instead, it is assumed that $m_w \geq 2$. As explained in the proof of Proposition 1 of the current paper, these assumptions can be replaced by reverse hazard rate dominance and $m_w \geq 1$. There are $n = m_s + m_w$ bidders in total. Finally, assume that $\overline{v}_s > \overline{v}_w > \underline{v}_s \geq \underline{v}_w$.

Preliminaries: Let $\varphi_i(b)$ denote bidder i's inverse bidding strategy. Consider the range of bids where all bidders are active. If bidder i with type v contemplates bidding in this range, his expected payoff is $(v-b)\prod_{j\neq i} F_j(\varphi_j(b))$, which is maximized where

$$\ln(v-b) + \sum_{j\neq i} \ln F_j(\varphi_j(b))$$

is maximized. Deriving the first order condition and imposing the equilibrium condition that $v = \varphi_i(b)$ produces

$$\sum_{j\neq i} \frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b}. \tag{14}$$

Summing (14) across all agents and subtracting (14) for agent i yields the system of differential equations in (7).

Proposition: Assume $m_s' \geq m_s \geq 2$, $m_w' \geq m_w \geq 2$ and $m_s' + m_w' > m_s + m_w$. Assume there is a binding reserve price in place, with $r \in (\underline{v}, \min\{\overline{v}_s, \overline{v}_w\})$. Then, $\widehat{v}' \leq \widehat{v}$ with $\widehat{v}' < \widehat{v}$ if $\widehat{v} < \overline{v}_s$. Consequently, if bid-separation occurs under (m_s, m_w) then it also occurs under (m_s', m_w'), i.e. as the number of bidders increases.

Proof. Fix m_s and consider a change in m_w. The proposition is trivially true if $\widehat{v} = \overline{v}_s$. Hence, assume $\widehat{v} < \overline{v}_s$. Since the relationship

$$\widehat{v} = \min\left\{\overline{v}_s, \frac{m_s}{m_s - 1}\overline{v}_w - \frac{1}{m_s - 1}\overline{b}_w\right\} \tag{15}$$

does not depend on m_w, it follows that $\widehat{v}' < \widehat{v}$ if and only if $\overline{b}_w' > \overline{b}_w$, where $\overline{b}_w'$ and $\overline{b}_w$ are the maximum bids of weak bidders in the setting with (m_s, m_w') and (m_s, m_w) bidders, respectively. Thus, assume by contradiction that $\overline{b}_w' \leq \overline{b}_w$.

The proof proceeds in two steps. First, inverse bidding strategies in the two settings are compared. In the second step, this comparison makes it possible to contradict the starting assumption that $\overline{b}_w' \leq \overline{b}_w$.

Consider first the case where the inequality is strict, or $\overline{b}_w' < \overline{b}_w$ and thus $\widehat{v}' > \widehat{v}$. Let $\gamma_i(b)$ denote the inverse bidding strategies with (m_s, m_w') bidders and let $\varphi_i(b)$ denote the inverse bidding strategies with (m_s, m_w) bidders. By assumption, $\gamma_i(\overline{b}_w') > \varphi_i(\overline{b}_w')$, $i = s, w$. Now reduce the bid from $\overline{b}_w'$ until the first point is reached (if one exists) for which $\gamma_i(b) = \varphi_i(b)$ but $\gamma_j(b) \geq \varphi_j(b)$, for some $i = s, w$, with $i \neq j$. If $i = w$, then from (7)

$$\begin{aligned} \frac{d}{db} \ln F_w(\gamma_w(b)) &= \frac{1}{m_s + m_w' - 1} \left[\frac{m_s}{\gamma_s(b) - b} - \frac{m_s - 1}{\gamma_w(b) - b} \right] \\ &< \frac{1}{m_s + m_w - 1} \left[\frac{m_s}{\varphi_s(b) - b} - \frac{m_s - 1}{\varphi_w(b) - b} \right] \\ &= \frac{d}{db} \ln F_w(\varphi_w(b)). \end{aligned}$$

However, this contradicts the fact that $\gamma_w > \varphi_w$ to the right of b (as b is the highest bid at which γ_w and φ_w intersects and $\gamma_w(\overline{b}_w') > \varphi_w(\overline{b}_w')$). Thus, assume instead that $\gamma_s(b) = \varphi_s(b)$ but $\gamma_w(b) \geq \varphi_w(b)$. Here, we wish to compare $\frac{d}{db} \ln F_s(\gamma_s(b))$ and $\frac{d}{db} \ln F_s(\varphi_s(b))$. If the former is strictly smaller than the latter, we obtain the same contradiction as above. On the other hand, it is easy to verify that

$$\frac{d}{db} \ln F_s(\gamma_s(b)) \geq \frac{d}{db} \ln F_s(\varphi_s(b))$$

at some b where $\gamma_s(b) = \varphi_s(b)$ but $\gamma_w(b) \geq \varphi_w(b)$ implies that

$$\frac{m_s}{\varphi_s(b) - b} - \frac{m_s - 1}{\varphi_w(b) - b} \leq 0,$$

which in turn means that $\frac{d}{db} \ln F_s(\varphi_s(b)) \leq 0$. However, this contradicts the equilibrium property that $\frac{d}{db} \ln F_s(\varphi_s(b)) > 0$ in the interior. In other words, there can be no intersection between γ_i and φ_i as b is reduced from $\overline{b}_w'$. Thus, we conclude that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \overline{b}_w']$, $i = s, w$.

The second case is $\overline{b}_w^{\prime} = \overline{b}_w$ and $\widehat{v}' = \widehat{v}$. Here, it can be shown that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \overline{b}_w^{\prime})$. A sketch is given next (with details available on request). By assumption, $\gamma_i(\overline{b}_w^{\prime}) = \varphi_i(\overline{b}_w^{\prime})$. Moreover,

$$\begin{aligned}
\frac{d}{db}\ln F_w(\gamma_w(b))_{|b=\overline{b}_w^{\prime}} &= \frac{d}{db}\ln F_w(\varphi_w(b))_{|b=\overline{b}_w^{\prime}} = 0 \\
\frac{d}{db}\ln F_s(\gamma_s(b))_{|b=\overline{b}_w^{\prime}} &= \frac{d}{db}\ln F_s(\varphi_s(b))_{|b=\overline{b}_w^{\prime}} = \frac{1}{m_s}\frac{1}{\overline{v}_w - \overline{b}_w^{\prime}}.
\end{aligned}$$

However, simple differentiation and tedious algebra can be used to prove that

$$\frac{d^2}{db^2}\ln F_s(\gamma_s(b))_{|b=\overline{b}_w^{\prime}} > \frac{d^2}{db^2}\ln F_s(\varphi_s(b))_{|b=\overline{b}_w^{\prime}}$$

and thus

$$\frac{d}{db}\ln F_s(\gamma_s(b)) < \frac{d}{db}\ln F_s(\varphi_s(b))$$

for b close to, but strictly below, $\overline{b}_w^{\prime}$. In other words, $\gamma_s(b) > \varphi_s(b)$ for b close to, but strictly below, $\overline{b}_w^{\prime}$. This property can then be used to establish that $\gamma_w(b) > \varphi_w(b)$ for b close to, but strictly below, $\overline{b}_w^{\prime}$. The argument from the first case ($\overline{b}_w^{\prime} = \overline{b}_w$) then applies to prove that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \overline{b}_w^{\prime})$.

The next step utilizes (14). Specifically, the above ranking of inverse bidding strategies implies that

$$\begin{aligned}
\frac{d}{db}\ln F_s(\gamma_s(b))^{m_s-1}F_w(\gamma_w(b))^{m_w^{\prime}} &= \frac{1}{\gamma_s(b) - b} \\
&< \frac{1}{\varphi_s(b) - b} \\
&= \frac{d}{db}\ln F_s(\varphi_s(b))^{m_s-1}F_w(\varphi_w(b))^{m_w}
\end{aligned}$$

for all $b \in (r, \overline{b}_w^{\prime})$. Equivalently

$$\frac{d}{db}\left[\ln\left(\frac{F_s(\gamma_s(b))}{F_s(v')}\right)^{m_s-1} F_w(\gamma_w(b))^{m_w^{\prime}}\right] < \frac{d}{db}\left[\ln\left(\frac{F_s(\varphi_s(b))}{F_s(\varphi_s(\overline{b}_w^{\prime}))}\right)^{m_s-1}\left(\frac{F_w(\varphi_w(b))}{F_w(\varphi_w(\overline{b}_w^{\prime}))}\right)^{m_w}\right].$$

The two terms in brackets coincide at $b = \overline{b}_w^{\prime}$, where they are both equal to zero. Since $r > \underline{v}$, both bracketed terms converge to finite values as $b \to r$. However, since the bracketed term on the left is flatter than its counterpart on the right, is must hold that

$$\ln\left(\frac{F_s(r)}{F_s(v')}\right)^{m_s-1} F_w(r)^{m'_w} > \ln\left(\frac{F_s(r)}{F_s(\varphi_s(\overline{b}'_w))}\right)^{m_s-1}\left(\frac{F_w(r)}{F_w(\varphi_w(\overline{b}'_w))}\right)^{m_w}$$

since $\gamma_i(r) = \varphi_i(r) = r$, $i = s, w$. Since $v' = \gamma_s(\overline{b}'_w) \geq \varphi_s(\overline{b}'_w)$ and $F_w(\varphi_w(\overline{b}'_w)) \leq 1$,

$$\left(\frac{F_s(r)}{F_s(v')}\right)^{m_s-1} F_w(r)^{m'_w} > \left(\frac{F_s(r)}{F_s(v')}\right)^{m_s-1} (F_w(r))^{m_w}$$

or $F_w(r)^{m'_w} > (F_w(r))^{m_w}$. However, since $F_w(r) \in (0,1)$ and $m'_w > m_w$, this is impossible. Hence, a contradiction to the assumption that $\overline{b}'_w \leq \overline{b}_w$ has now been obtained.

Next, fix m_w instead and let m_s increase to m'_s. Again, the proposition is trivially true if $\widehat{v} = \overline{v}_s$. Hence, assume $\widehat{v} < \overline{v}_s$. Note that the relationship in (15) depends on m_s. In fact, (15) implies that if $\widehat{v}' \geq \widehat{v}$ then $\overline{b}'_w < \overline{b}_w$ is necessary. Assume by contradiction that $\widehat{v}' \geq \widehat{v}$ and note now that $\gamma_s(\overline{b}'_w) = \widehat{v}' > \widehat{v} = \varphi_s(\overline{b}_w) > \varphi_s(\overline{b}'_w)$ and likewise that $\gamma_w(\overline{b}'_w) = \overline{v}_w = \varphi_w(\overline{b}_w) > \varphi_w(\overline{b}'_w)$. Hence, $\gamma_i(\overline{b}'_w) > \varphi_i(\overline{b}'_w)$, $i = s, w$. Following the same steps as above then yields the contradiction.

Thus, it has now been established that if either m_s or m_w increases then $\widehat{v}' \leq \widehat{v}$ with $\widehat{v}' < \widehat{v}$ if $\widehat{v} < \overline{v}_s$. Since increases in m_s and m_w move the equilibrium in the same direction, the same conclusion holds if both m_s and m_w increase at the same time. This concludes the proof. ■