On Technological Heterogeneity in Contests*

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Abstract

There are many facets to heterogeneity in contests. The typical approach is to focus on differences in preferences or narrowly defined abilities. This paper instead considers a contest model in which performance is stochastic and described by identity-dependent distributions that are so different that agents are unable to replicate the distribution of another's performance. As a result of this technological heterogeneity, the favorite, i.e. the agent who is most likely to win the contest, may no longer view actions as strategic complements. Compared to other contest models, this in turn changes comparative statics, policy implications, and the effects of precommitment.

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1 Introduction

Agents in contests can be heterogenous for a wide variety of reasons. College applicants may be from different socioeconomic backgrounds and have access to varying resources as they strive to improve their GPA or prepare for the SAT. Candidates for promotion may have distinct educational backgrounds and professional experiences. Competing lobbyists or litigators represent different organizations or clients, each with their own resources and objectives. Firms in R&D races have different patents and institutional knowledge to build on. The resulting heterogeneity can manifest itself in a multitude of ways. To name a few, agents may value the prize differently, their budgets may not be the same, and their ability to effect an improvement in their performance in the contest, or the cost of doing so, may be poles apart. These are the kinds of heterogeneity that are typically studied in the literature. In equilibrium, the favorite, i.e. the agent most likely to win the contest, tends to be the agent whose characteristics are in some way stronger or more advantageous.

Although the agent's action is productive, his performance in the contest is still likely to be stochastic. The common approach is to assume that the nature or structure of the noise is the same for all agents. However, this is restrictive. For instance, one agent may generally be less error prone than another, or he may become less susceptible to errors when he works harder while another may become more erratic. This paper pursues a contest model – the mixture model – that allows idiosyncrasies of this kind. The extra layer of heterogeneity may override the effects of the characteristics mentioned earlier. For instance, the favorite may care less about the prize, yet benefit from being less error-prone than his competitor.

There are now two distinct drivers that determine who is the favorite, each with potentially different implications. A standard model like the Tullock contest predicts that the favorite (underdog) views actions as strategic complements (substitutes) in two-player contests. However, the opposite may apply in the mixture model. The practical implication is that it becomes important to understand the root cause of one agent's emergence as the favorite, because comparative statics and policy implications may be different. For instance, if a contest organizer wants to spur both agents to work harder, then one option is to increase the prize for the underdog, try to subsidize his effort, or the like. This incentivizes the underdog to put in more effort. If the favorite views actions as strategic complements, then he responds by working harder too. However, if it is the underdog that views actions as strategic complements, then it may be better to incentivize the favorite. In other words, the mixture model invites a reexamination of what it means and entails to be the favorite or the underdog.

In a similar vein, having identified a second dimension of heterogeneity raises general questions about how best to fit certain policy interventions and design choices into a contest model. For instance, is improving the quality of secondary schooling in an impoverished area simply equivalent to lowering the effected student's cost of effort, or is it better described as a more nuanced alteration of the very technology that he has access to when he prepares for the SAT? Similarly, when a salesman in a promotion contest is assigned a particularly loyal set of customers, it is reasonable to expect that it reduces the variance of his performance, but this is more subtle than a mere change in the cost function. While these questions are not central to the paper, an example shows that under some, possibly rare, circumstances, improving the technology of an agent may lower his chances of winning the contest in equilibrium.

To better understand and motivate the kind of heterogeneity studied here, it is useful to begin by reexamining a common justification of the popular contest success function introduced by Tullock (1975, 1980). Hirschleifer and Riley (1992), Fullerton and McAfee (1999), and Jia (2008) show that it can be microfounded as a contest with stochastic performance. However, these microfoundations assume that the performance of all agents are described by the *same* parameterized distribution function. Differences in how actions influence the parameter are permitted, through differences in the so-called impact function. This is a rather restrictive and one-dimensional form of technological heterogeneity. For instance, if two agents have the same expected performance, then their distributions must be exactly *identical*. Thus, heterogeneity in ability is narrowly defined as simply measuring how easy or costly it is to change a parameter in a distribution function that is common to all.

Consider a promotion contest in which the engineer with the best idea of how to improve a product is promoted. One engineer is unimaginative but consistent and well trained, while the other is less conventional and thinks outside the box. Even if their ideas are equally good in expectation, the performance of the second engineer is likely to be more erratic. Stated differently, their performance is not identically distributed. The mixture model incorporates a more flexible expression of asymmetric skills of this form, by allowing different agents to have access to different parameterized distribution functions. In view of current terminology, the term ability will be used to refer to how easy or costly it is to manipulate the parameter. The term technological heterogeneity is used when distribution functions do not coincide even if the parameters are the same. Two agents with the same preferences and abilities who take the same action and win with the same probability will have the same expected utility. However, if their technologies are heterogeneous, then they are unlikely to win with the same probability in the first place.

As in other contest models, the mixture model allows agents to have heterogeneous valuations, impact functions, and cost functions.[1] The distribution of performance is a mixture distribution with two components and endogenous weights that are determined by the agent's action and impact function. The higher his action, the more likely it is that his performance is drawn from a good rather than a bad distribution. The mixture components may be asymmetric across agents. The agents have heterogeneous technologies when this is so.

One way to think about the model is that the bad mixture component represents the agent's innate level of competency or aptitude (from minimal effort) and the good mixture component his competency or learned and acquired skill if he realizes his full potential (maximum effort). The impact function measures how far the agent has come in his journey from innate ability to full potential, or from floor to ceiling. Then, the distribution of performance is a convex combination of the two extremes, with the weights determined by how far the agent has decided to journey. The expected performance of two agents at the same step in their journey can be different because their innate ability and full potential may differ. As in the engineering example, it is also possible that their expected performance is the same, but that the variance differ or indeed that it changes with the action in different ways. It turns out that technological heterogeneity has implications that cannot be duplicated simply by allowing impact functions or preferences and abilities to be heterogeneous.[2]

The reason why heterogeneous technologies have different consequences than heterogeneous preferences and abilities is that they influence the reaction functions in different ways. To begin, in two-player Tullock contest with complete information, the

[1] A companion paper, Kirkegaard (2022), presents a more general version of the model that permits incomplete information about these characteristics. However, that paper utilizes the mixture model to consider larger contests and focuses primarily on the case of homogenous technologies.

[2] See Section 6 for a discussion of a few limited ways in which technological heterogeneity can be incorporated into Tullock contests and rank-order tournaments. The Tullock contest imposes so much structure that the technological heterogeneity that it allows can be duplicated by changing the impact function. The rank-order tournament is more flexible.

reaction functions are hump-shaped. Starting from a symmetric setting, an agent's reaction function shifts outwards when his preferences and impacts become "stronger". The equilibrium action profile is thus at a place where the stronger agent, who is now the favorite, has an upwards sloping reaction function and the weaker agent a downwards sloping reaction function. In words, the favorite (underdog) views actions as strategic complements (substitutes) in equilibrium.

In a two-player mixture contest, changes in preferences or impacts also shift the reaction function. However, the "shape" of the reaction function is determined *solely* by the relationships between the mixture components. The two reaction functions are monotonic and they slope in opposite directions. Thus, one agent views actions as strategic complements and the other sees them as strategic substitutes in a global sense. Which agent is which does not depend on preferences or impacts, and it is entirely possible that it is the underdog who views actions as strategic complements. Consequently, the comparative statics may differ from other contest models. Likewise, extensions to contests with sequential moves may lead to different conclusions. Hence, a central question of this paper is what makes an agent view actions as complements or substitutes in two-player contests, and how this relates to his equilibrium status as favorite or underdog. It is shown that both the "strength" and the "spread" of the mixture components matter.

Aside from the role of the technologies in determining who views actions as strategic complements, there are two entangled effects when it comes to deciding who is the favorite. First, even if the impacts are held fixed across agents, heterogeneous technologies tend to imply that one agent is more likely to win than another. Second, incentives are not the same across agents. The reason is that the return to a marginal increase in an agent's action is measured by the increase in the winning probability that is entails, yet this is likely to differ from agent to agent when technologies are heterogenous. Thus, it is possible that the favorite has worse incentives and works less hard, but wins more often due to a technological advantage.[3]

To illustrate all these effects, return to the engineering example and imagine that all mixture components are normal distributions. Further, the two bad components have the same (low) mean and the two good components have the same (high) mean.

[3] In the Tullock contest, it is also possible that the agent with the lower action is the favorite, provided that his impact function is more advantageous. This effect is present also in the mixture model, but technological heterogeneity adds an extra effect.

When switching from the bad to the good component, the variance declines for the unimaginative engineer but increases for the unconventional engineer. Then, the latter views actions as complements. He is more tempted to bet on outliers when his competitor works harder, but this requires working harder himself. However, this does not on its own imply that he works harder than his competitor in equilibrium. For instance, it is intuitive that equilibrium actions are low if valuations are low. Then, viewing actions as complements, the unconventional engineer is further dissuaded from working hard by the fact that his competitor is not working hard. At higher valuations, this may be reversed and the unconventional engineer may work harder than the unimaginative engineer. However, even in this case the chance of bad outliers means that he may nevertheless end up as the underdog.

In summary, comparative statics and policy recommendations in contests generally rely on identifying which agent views actions as strategic complements. The mixture model is sufficiently tractable that it allows a thorough examination of the role of technological heterogeneity in determining who this agent is.

2 The model

This paper pursues a version of the mixture model with two-players, complete information, and concave payoff functions. A companion paper, Kirkegaard (2022), presents a more general version that allows for several players, incomplete information about preferences and abilities, and non-concave payoffs. Incomplete information is easily handled in the mixture model, but is assumed away in the present paper to simplify notation and focus on heterogeneous technologies.

There is a single prize. Each agent is characterized in part by his preferences and abilities, and in part by his technology. Starting with the first and more familiar part, agent i has non-negative valuation v_i and an action set A_i that is a non-empty and compact interval. The agent's impact function is $p_i(a_i) \in [0,1]$ and his cost function is $c_i(a_i)$, $a_i \in A_i$. Assume that $p_i'(a_i), c_i'(a_i) > 0$ and that $p_i''(a_i) \leq 0$ and $c_i''(a_i) \geq 0$, with at least one strict inequality. The agent's payoff is $v_i - c_i(a_i)$ if he wins with action a_i, and $-c_i(a_i)$ if he loses. Participation in the contest is mandatory. The winner of the contest is the agent with the best performance.

Agent i's performance is a random variable, X_i, which is unidimensional and with a distribution that is determined in part by the agent's action and impact function.

Upon taking action a_i, agent i's performance follows the mixture distribution

$$F_i(x_i|a_i) = p_i(a_i)H_i(x_i) + (1 - p_i(a_i))G_i(x_i). \tag{1}$$

The mixture components H_i and G_i are themselves distribution functions. They are assumed to be atomless and have densities h_i and g_i, respectively. All mixture components are independent of each other and of actions. They also have the same support, which in turn implies that the performance of the two agents have the same support for all possible action profiles. Hence, for any action profile, any agent always has a chance of winning the contest. Ties can be ignored because the distributions are atomless. Finally, assume that $H_i < G_i$ on the interior of the support. In other words, H_i first order stochastically dominates G_i, which therefore implies that $F_i(x_i|a_i)$ improves in a first-order stochastic sense when the agent works harder. Let μ_{H_i} and μ_{G_i} denote the expected values of H_i and G_i respectively, with $\mu_{H_i} > \mu_{G_i}$.

There are two sources of heterogeneity in the model. Inspired by the existing literature, the agents will be said to have homogeneous *preferences and abilities* if they have the same actions sets, valuations, cost functions, and impact functions. Otherwise, their preferences and abilities are heterogenous. The two agents will be said to have homogenous *technologies* if they have symmetric mixture components, or $H_i = H_j$ and $G_i = G_j$. Otherwise, their technologies are heterogeneous.

Given the action profile, agent i's probability of winning the contest is

$$\begin{aligned} q_i(a_i, a_j) &= \int F_j(x|a_j) f_i(x|a_i) dx \\ &= \int \left(p_j(a_j)H_j(x) + (1 - p_j(a_j))\, G_j(x)\right) \left(p_i(a_i)h_i(x) + (1 - p_i(a_i)\, g_i(x)\right) dx, \end{aligned}$$

where it is understood that $j \neq i$. Note that $q_i(a_i, a_j)$ is a contest success function (CSF), in the sense that it takes the action profile as an input and outputs a winning probability. Given the action profile, agent i's expected utility is

$$u_i\,(a_i, a_j) = v_i q_i\,(a_i, a_j) - c_i(a_i).$$

The assumptions imposed so far implies that $F_i(x_i|a_i)$ is convex in a_i. Invoking a standard result from the moral hazard literature that flows from Rogerson (1985), the expectation of any function that is increasing in x – such as $F_j(x|a_j)$ – is therefore concave in a_i. In other words, $q_i(a_i, a_j)$ is concave in a_i, and strictly so if $p_i''(a_i) < 0$. Formally, this can be confirmed through integration by parts. Hence, $u_i(a_i, a_j)$ is strictly concave in the agent's action when $v_i > 0$. The best response to a_j is therefore unique and any equilibrium is in pure strategies.

Since A_i is compact, a best response always exists. The best response is increasing in a_j if $q_i(a_i, a_j)$ is supermodular, or $\frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j} > 0$, and decreasing in a_j if $q_i(a_i, a_j)$ is submodular, or $\frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j} < 0$. In the first case, agent i considers actions to be strategic complements and in the second case he views actions as strategic substitutes. Indeed, the sign of

$$\frac{\partial^2 q_i(a_i, a_j)}{\partial a_i \partial a_j} = p_1'(a_1) p_2'(a_2) \int (H_j(x) - G_j(x)) (h_i(x) - g_i(x)) \, dx, \tag{2}$$

is independent of the action profile. This implies that agent i's best response function is monotonic, i.e. it is either globally increasing, globally decreasing, or horizontal. Hence, it does not take the hump-shape that is familiar from the Tullock contest. Note also that the sign of (2) is the same for all increasing impact functions. In other words, whether the agent considers actions to be complements or substitutes depends only on the mixture components and their interaction.

Moreover, since winning probabilities sum to one for all action profiles, it holds that $\sum_{i=1}^{2} \frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j} = 0$. Hence, either (2) takes the opposite sign for the two agents, or it is zero for both agents. In the former case, one agent views actions as strategic complements and the other sees them as strategic substitutes. In the latter case, there is no strategic interaction and equilibrium is therefore in strictly dominant strategies.

As a special case, assume that technologies are homogenous. Then, (2) coincide for the two agents and it must therefore be zero. Hence, equilibrium is in strictly dominant strategies. Examples will later demonstrate that there are knife-edge cases where this may also hold when technologies are heterogenous. When equilibrium is not in dominant strategies, one best response function slopes upwards and the other slopes downwards. Hence, they cross exactly once. Thus, equilibrium is unique. Proposition 1 summarizes the discussion so far.

Proposition 1 *There is a unique Nash equilibrium of any two-player mixture contest. If technologies are homogenous, then the equilibrium is in strictly dominant strategies. If equilibrium is not in strictly dominant strategies, then one agent views actions as strategic complements and the other sees them as strategic substitutes.*

To derive equilibrium, and following Kirkegaard (2022), it is useful to expand $q_i(a_i, a_j)$, yielding

$$q_i(a_i, a_j) = t_i\left(p_j(a_j)\right) + p_i(a_i)k_i\left(p_j(a_j)\right)$$

where

$$t_i\left(p_j\right) = \int \left(p_j H_j(x) + \left(1 - p_j\right) G_j(x)\right) \times g_i(x)dx$$

is the base probability that agent i wins the contest with a draw from G_i and

$$k_i\left(p_j\right) = \int \left(p_j H_j(x) + \left(1 - p_j\right) G_j(x)\right)\left(h_i(x) - g_i(x)\right)dx$$

is a measure of the return to agent i's action since it measures the increase in winning probability when agent i's performance is drawn from the good component rather than the bad components. Since H_i first-order stochastically dominates G_i, $k_i\left(p_j\right) > 0$. The reason that t_i and k_i are expressed in terms of p_j rather than a_j directly is that $p_j \in [0, 1]$ for any mixture model, which makes it easier to bound and describe these functions without invoking details about preferences and abilities. This turns out to be of expositional convenience later on when comparative statics are examined.

The expansion of $q_i(a_i, a_j)$ is helpful because the first term in the expansion is independent of agent i's action. Hence, given a_j, agent i's problem is to maximize

$$U_i\left(a_i, a_j\right) = v_i p_i(a_i)k_i\left(p_j(a_j)\right) - c_i(a_i)$$

with respect to a_i. Rewriting,

$$k_i(p_j) = \int G_j(x)\left(h_i(x) - g_i(x)\right)dx + p_j \int \left(H_j(x) - G_j(x)\right)\left(h_i(x) - g_i(x)\right)dx,$$

or, with the appropriate definitions of the coefficients,

$$\begin{aligned} k_1(p_2) &= \kappa_0 + \kappa_1 p_2 \\ k_2(p_1) &= \tau_0 + \tau_1 p_1. \end{aligned}$$

Note that $p_1'(a_1)p_2'(a_2)\left(k_1'(p_2) + k_2'(p_1)\right) = \sum_{i=1}^{2} \frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j} = 0$, or $k_1'(p_2) + k_2'(p_1) = 0$.

Example 1: Assume that $p_i(a_i) = \sqrt{a_i}$, $c_i(a_i) = a_i$, $A_i = [0, 1]$, $i = 1, 2$. Assume that $v_i k_i(0) - 2 \leq 0$ and $v_i k_i(1) - 2 \leq 0$ for $i = 1, 2$. This means that valuations are not too high and ensures that best responses are interior whenever $v_i > 0$. From the first-order condition, agent i's best response to a_j is $a_i = \left(\frac{1}{2}v_i k_i(p_j(a_j))\right)^2$. This implies that $p_i(a_i) = \frac{1}{2}v_i k_i(p_j(a_j))$. Thus, in equilibrium, (a_2, a_2) solves the system

$$\begin{aligned} p_1(a_1) &= \frac{1}{2}v_1 \left(\kappa_0 + \kappa_1 p_2(a_2)\right) \\ p_2(a_2) &= \frac{1}{2}v_2 \left(\tau_0 + \tau_1 p_1(a_1)\right) \end{aligned}$$

which is a linear system in p_1 and p_2. With this transformation, the solution is

$$(p_1^*, p_2^*) = \left(\frac{2\kappa_0 v_1 + \kappa_1 \tau_0 v_1 v_2}{4 - \kappa_1 \tau_1 v_1 v_2}, \frac{2\tau_0 v_2 + \kappa_0 \tau_1 v_1 v_2}{4 - \kappa_1 \tau_1 v_1 v_2}\right)$$

and the equilibrium action profile is therefore $(a_1^*, a_2^*) = \left((p_1^*)^2, (p_2^*)^2\right)$. ▲

A key property of the two-player mixture model is that it is solely the technologies that determine which agent views actions as strategic complements and which agent sees them as strategic substitutes. It has nothing to do with valuations, impact functions, or cost functions, i.e. the factors that are traditionally considered in the contest literature. Nor does the action profile matter. In comparison, in two-player Tullock contests the "favorite" (the agent who is more likely to win in equilibrium) has an upward sloping best response function and the "underdog" a downwards sloping best response function in a neighborhood around the equilibrium action profile.

The next two sections studies comparative statics. There are now two dimensions to study. As always, the consequences of changes in preferences and impacts can be analyzed. However, the mixture model also invites an examination of the consequences of changes in the performance technologies themselves. Comparative statics in this vein are considered first, since this is new. An important finding is that it is not necessarily the favorite who views actions as strategic complements.

3 Comparative statics: Technologies

This section studies the role of the technologies in determining who views actions as strategic complements, who faces the stronger incentives, and who is the favorite.

The properties of k_1 and k_2 are key to these questions, yet k_1 and k_2 depend in a potentially complicated way on the interaction of the four mixture components (henceforth referred to simply as components). Thus, the analysis proceeds by studying two environments that are more specialized. These capture two distinct ways in which the components can differ. The first focuses on how "strong" any given component is, while the second concentrates on how "spread out" it is. To illustrate, assume the component's distribution belongs to some location-scale family of distributions. Shifting the location then translates into changing the strength of the component, while manipulating the scale translates into changing the spread of the component.

As alluded to in the introduction, the variance of agent i's performance depends on his action. Let $\sigma^2_{H_i}$ and $\sigma^2_{G_i}$ denote the variance of H_i and G_i, respectively. Then, the variance of the agent's ex ante performance, given a_i, can be shown to equal

$$\mathrm{Var}\left[X_i|a_i\right] = p_i(a_i)\sigma^2_{H_i} + (1 - p_i(a_i))\,\sigma^2_{G_i} + (1 - p_i(a_i))p_i(a_i)(\mu_{H_i} - \mu_{G_i})^2. \quad (3)$$

Thus, the variance changes with the agent's effort. If $\sigma^2_{H_i} = \sigma^2_{G_i}$, then the variance is maximized when $p_i(a_i) = \frac{1}{2}$. Otherwise, the variance may be monotonic in a_i. When technologies are heterogenous, the way in which $\mathrm{Var}[X_i|a_i]$ evolves with a_i is typically different from agent to agent as well, even if impact functions are the same.

3.1 On the strength of the mixture components

Assume that all components belong to the same parameterized family of distributions, $F^C(x|\eta)$, where the exogenous parameter η measures the strength of the component. Assume that $F^C(x|\eta)$ and its density $f^C(x|\eta)$ are differentiable in η. Assume that $F^C_\eta(x|\eta) < 0$ for all interior x, where the subscript refers to the partial derivative. This means that low outcomes are less likely the higher η is, thus explaining why η is a measure of strength. More concretely, the components are described by

$$H_i(x) = F^C(x|\alpha_i),\, G_i(x) = F^C(x|\beta_i),\, i = 1, 2,$$

where α_i and β_i are exogenously given and satisfy $\alpha_i > \beta_i$ to ensure that H_i first order stochastically dominates G_i. The difference between α_i and β_i measures how much better agent i's good component is compared to his bad component. Comparing components across agents, agent 1 is said to have a *stronger technology* than agent 2 if $\alpha_1 \geq \alpha_2$, $\beta_1 \geq \beta_2$, and at least one of the inequalities is strict. In this case, agent 1's good and bad components first order stochastically dominates agent 2's good and bad components, respectively. Thus, agent 1's expected performance is higher than agent 2's, for comparable impacts. Agent 1 is said to have a *more sensitive technology* than agent 2 if $\alpha_1 \geq \alpha_2 > \beta_2 \geq \beta_1$ and either $\alpha_1 > \alpha_2$ or $\beta_2 > \beta_1$, or both. Here, agent 1's expected performance changes more rapidly with a change in the impact.

For a specific example, assume that $F^C(x|\eta) = F_0(x)^\eta$, where F_0 is some "parent" distribution and η is restricted to be strictly positive. Technologies of this form will be referred to as *power technologies* since each component raises the parent distribution to some power. Such technologies are particularly easy to work with because it turns out that the parent distribution does not influence $t_i(p_j)$ or $k_i(p_j)$ at all. What matters is solely how the strengths of the components relate to each other. Thus, it is possible to obtain more complete results for power technologies.[4]

The family F^C is henceforth assumed to have satisfy an additional regularity property. Specifically, $-F^C_\eta$ is log-supermodular in x and η. This assumption is introduced, and discussed in great detail, by Chade and Swinkels (2020). Although they use it in a different context and for very different purposes, it is also useful here because it is ideally suited for comparing different parameter values. Chade and Swinkels (2020) show that the assumption is equivalent to

$$\frac{\partial}{\partial x}\frac{F^C(x|\beta_1) - F^C(x|\alpha_1)}{F^C(x|\beta_2) - F^C(x|\alpha_2)} \geq 0 \text{ when } \alpha_1 \geq \alpha_2 \text{ and } \beta_1 \geq \beta_2, \tag{4}$$

keeping in mind that it has already been assumed that $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$. The assumption is satisfied for power technologies. Extending a point made in Chade and Swinkels (2020), the assumption also hold if F^C is the location-scale family, or $F^C(x|\eta) = F_0(\frac{x-\eta}{\sigma})$, whenever the parent distribution F_0 has a log-concave density. To simplify the exposition of results, the assumption will be strengthened slightly to assume that $-F^C_\eta$ is strictly log-supermodular in x and η and that the ratio in (4) is strictly increasing in the interior of the support when $\alpha_1 \geq \alpha_2$, $\beta_1 \geq \beta_2$, and at least one of the inequalities is strict, i.e. when agent 1 has the stronger technology.

In the present context, $F^C(x|\beta_1) - F^C(x|\alpha_1)$ in (4) can be interpreted as the reduction in the probability of a performance worse than x if the draw is made from agent 1's good component rather than his bad component, or equivalently the improvement in the probability that the performance is better than x. Note that this

[4]It can be verified that if $H_i(x) = 1-(1-F_0(x))^{1/\alpha_i}$, $G_i(x) = 1-(1-F_0(x))^{1/\beta_i}$, $\alpha_i > \beta_i > 0$, $i = 1, 2$, then $t_i(p_j)$ and $k_i(p_j)$ are exactly the same as for power technologies. This isomorphism holds only in two-player contests, however. As an example, if F_0 is an exponential distribution with mean one, then H_i and G_i are exponential distributions with means of α_i and β_i, respectively.

is a measure of the return to extra effort, as extra effort makes a draw from the good component more likely. The condition thus means that if both agents work harder, agent 1's return is relatively speaking better compared to agent 2's return the higher the performance is. A better performance is more relevant because it is more likely to be a winning performance. Therefore, it is in agent 1's interest to work harder when agent 2 does so. In other words, agent 1 views actions as strategic complements when he has the stronger technology.

Proposition 2 *Agent 1 views actions as strategic complements if he has the stronger technology. For power technologies, agent 1 views actions as strategic complements if and only if $\alpha_1\beta_1 > \alpha_2\beta_2$.*

Figure 1 illustrates Proposition 3 when $\alpha_2 = 2$ and $\beta_2 = 1$ are fixed but β_1 and α_1 vary. Since $\alpha_1 > \beta_1$, only the area above the 45° line is relevant. Agent 1 views actions as strategic complements to the north-east of (β_2, α_2) and as strategic substitutes to the south-west of this point. It is natural to conjecture that the closer (β_1, α_1) is to the former region, the more likely it is that agent 1 views actions as strategic complements. The proposition confirms this for the case of power technologies, since the higher α_1 or β_1 is, he more likely it is that the condition that $\alpha_1\beta_1 > \alpha_2\beta_2$ is satisfied. In Figure 1, the downwards sloping line is the level curve where $\alpha_1\beta_1 = \alpha_2\beta_2$. Thus, this curve delineates exactly when agent 1 views actions as complements or substitutes in the case of power technologies. If $\alpha_1\beta_1 = \alpha_2\beta_2$, then there is no strategic interaction and equilibrium is in strictly dominant strategies. This clearly applies if technologies are homogenous across agents, but it may also hold when technologies are heterogeneous.

Whether agent i considers actions to be strategic complements or substitutes depends on the sign of $k_i'(p_j)$. However, it is level of $k_i(p_j)$, not its slope, that determines how strong agent i's incentives are. Thus, it is not necessarily the case that the agent who views actions as strategic complements has the strongest incentives. Indeed, due to the differences between $k_1(p_2)$ and $k_2(p_1)$, the two agents face different incentives even if they have homogeneous preferences and abilities. For instance, as β_i approaches α_i, agent i's bad component becomes indistinguishable from the good component. Thus, there is little reason to take costly action to make the good component more likely. In other words, the agent's incentives disappear, even if his technology is more likely to be the strongest. In fact, the argument suggest that when one agent has a more sensitive technology than the other, then the former has

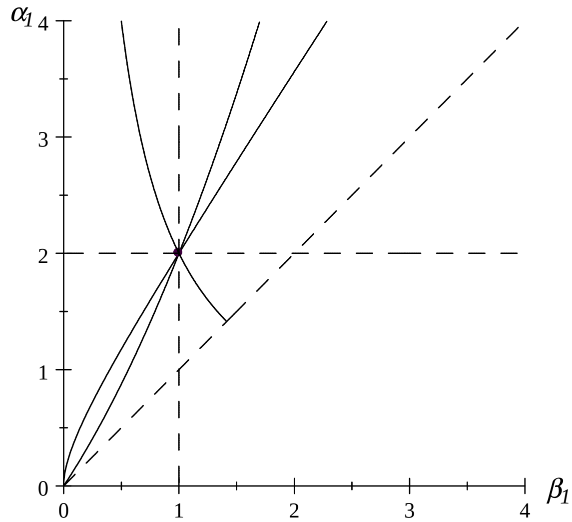

Figure 1: For power technologies, agent 1 views actions as strategic complement (substitutes) above (below) the downwards sloping curve. He has globally stronger (weaker) incentives to the left (right) of the two solid upwards sloping curves.

stronger incentives. The next result formalizes this intuition.

To robustly compare incentives, it is natural to ask under which conditions it holds that

$$\min_{p_2 \in [0,1]} k_1(p_2) \geq \max_{p_1 \in [0,1]} k_2(p_1). \tag{5}$$

If the agents have homogeneous preferences and abilities – regardless of what those preferences and abilities are – then (5) implies that agent 1's incentive to prove effort is no lower than those of agent 2. Thus, in the following, agent 1 is said to have *globally stronger incentives* than agent 2 if (5) applies.

Proposition 3 *Agent 1 has globally stronger incentives if he has the more sensitive technology. For power technologies, agent 1 has globally stronger incentives if*

$$\alpha_1 - \alpha_2 \geq \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{2\beta_1\beta_2}\left(\beta_1 - \beta_2\right) \text{ and } \alpha_1 - \alpha_2 \geq \frac{2\alpha_1\alpha_2}{\alpha_1\alpha_2 + \beta_1\beta_2}\left(\beta_1 - \beta_2\right).$$

Holding fixed α_2 and β_2, these conditions are more likely to hold the larger α_1 is and the smaller β_1 is.

Returning to Figure 1, agent 1 has the globally stronger incentives to the north-west of (β_2, α_2) and the weaker incentives to the south-east. Thus, it is natural to conjecture that the closer (β_1, α_1) is to the former region, the more likely it is that agent 1 has the stronger incentives. This is confirmed in the case of power technologies, where the two conditions in the proposition are more likely to hold the larger α_1 is and the smaller β_1 is. In Figure 1, each of the two upwards sloping lines describe the set of points where one of the conditions holds with equality. Agent 1 thus has stronger incentives at any point that is to the left of both curves. By combining Propositions 2 and 3, it is evident that in the region where agent 1 views actions as strategic complements, he may or may not have the globally stronger incentives.

If the two agents have homogeneous preferences and abilities, then agent i takes the higher action if he has the globally stronger incentives. However, since technologies are heterogenous, this does not necessarily mean that agent i wins with a higher probability. Moreover, even for power technologies, Proposition 3 is silent about the small area between the two solid curves in Figure 1. Here, neither agent has globally stronger incentives, and which agent works harder in equilibrium therefore depends on the specifics of the preferences and abilities. Example 2 builds on Example 1 to show that the favorite may or may not view actions as strategic complements.

Example 2: Begin with the setting in Example 1, but assume $v_1 = v_2 = 2$. Hence, agents have homogenous preferences and abilities. However, agents have heterogeneous power technologies, and $\alpha_2 = 2$ and $\beta_2 = 1$. It is then possible to calculate precisely when $a_1 = a_2$ in equilibrium. The curves in Figure 1 are reproduced as dashed curves in Figure 2. The solid upwards-sloping curve traces out when $a_1 = a_2$.[5] In comparison, the solid non-monotonic curve (valid only above the 45° line) describes when $q_1(a_1, a_2) = \frac{1}{2}$ in equilibrium, with $q_1(a_1, a_2) > \frac{1}{2}$ outside or above the curve. Note that the agent who wins most often may or may not be the agent who views actions as strategic complements. Since the curve in question has an upwards-sloping segment, agent 1 may win less often in equilibrium if α_1 and β_1 increase at the same time, even though this implies that his technology becomes stronger.

Consider two agents who are preparing to apply to college. To do so, they are investing effort into improving their GPA at their respective secondary schools. However, due to differences in socioeconomic backgrounds, they do not have access to the

[5] Appendix B gives a fuller account of how best-response functions depend on the parameters, and what inferences can be made concerning a_1 and a_2 without imposing assumptions on preferences. In particular, at any point in the parameter space, it is possible to characterize how k_1 and k_2 change with α_1 and β_1, which in turn makes it possible to sign the change in the equilibrium value of either a_1 or a_2. The change in the other action is ambiguous and depends on preferences.

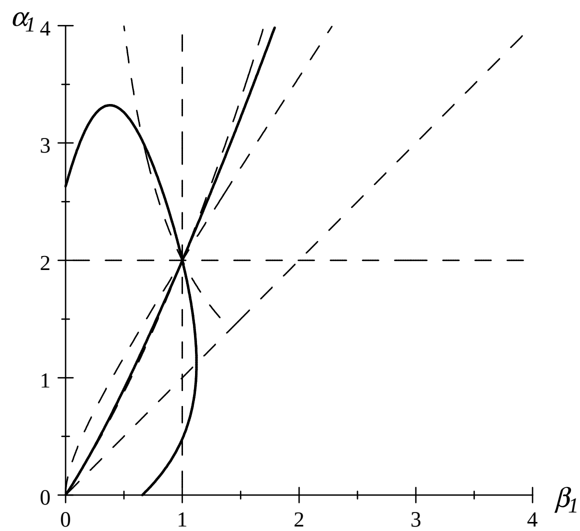

Figure 2: Equilibrium actions and winning probabilities.

same quality of secondary schooling. A public intervention that improves the quality of schooling for agent 1 can be modelled as an increase in α_1 and/or β_1. The model cautions that there are parameter constellations for which such an improvement distorts incentives in a way that ultimately leads agent 1 to be less likely to win the contest. Similarly, an affirmative action policy that scales up the performance of agent 1 before comparing him to agent 2 amounts to changing the distribution of his final "score". Depending on how this is implemented, it may be captured by an increase in α_1 and β_1. See Kirkegaard (2021) for an analysis of how to optimally design the contest rules in complete information contests with stochastic performance. ▲

In Figure 2, the part of the parameter space in which agent 1 views actions as strategic complements can be further divided into three smaller regions. First, $(\beta_1, \alpha_1) = (1, 3)$ is in a region where agent 1 has globally stronger incentives and is the favorite. Second, the point $(\beta_1, \alpha_1) = (2, 3)$ is in a region where agent 1 has globally weaker incentives but is nevertheless still the favorite. Here, his bad component is stronger than that of agent 2, and he can therefore rely on his superior base performance to maintain his status as the favorite in equilibrium. Third, the point $(\beta_1, \alpha_1) = (\frac{17}{20}, \frac{5}{2})$ belongs to a small region in which agent 1 has globally stronger incentives but is the underdog. In this case, he works harder than agent 2, but not hard enough to compensate for the fact that his base performance is poor.

3.2 On the spread of the mixture components

In the contests studied above, a change in the parameter changes the expected value of the component and therefore of the agent's performance for any fixed action. In the following analysis, the expected value of a given component is held fixed and focus is instead on the consequences of a change in its spread or variability.

Assume that all components have symmetric densities and that their supports are the entire real line. The good components have the same mean (and median), or $\mu_{H_1} = \mu_{H_2} = \mu_H$, but not necessarily the same distribution. For instance, H_1 and H_2 might belong to the same (symmetric) location-scale family, $H_i(x) = H_0(\frac{x-\mu_H}{\sigma_{H_i}})$, but have different scale or variance. The obvious example is when they are both normally distributed, with the same mean but different variance. However, it is not necessary that H_1 and H_2 belong to the same family of distributions. Similarly, the bad components have the same mean, or $\mu_{G_1} = \mu_{G_2} = \mu_G$, with $\mu_G < \mu_H$.[6]

By construction, if the two agents take actions that lead to the same impact, then their expected performance is the same. However, the spread of their performance, and how it depends on their action and impact, need not be the same.

The mixture distribution describes a compound lottery in which the agent draws from one of two components. There are thus four possible combinations across agents, in the sense that they can both draw from their respective good components, their respective bad components, or from "opposite" components. It is useful to consider the probability of winning conditional on each of the four cases. In the first two cases, components are matched across agents, and in the last two cases they are mismatched.

Recall that agent i considers actions to be strategic complements if $q_i(a_i, a_j)$ is supermodular. An increase in a_j increases the chance that both agents draw from their good components, and an increase in a_i only accelerates that increase. Similarly, an increase in a_j decreases the probability of a match of bad components, and an increase in a_i further decreases that probability. Hence, matched components push in the direction of making $q_i(a_i, a_j)$ supermodular. The mismatched components push in the other direction, towards submodularity. The question is which effect dominates for each agent. The discussion of this issue is made easier by letting X_{H_i} and X_{G_i}

[6] In this specification it is not necessary that H_i first order stochastically dominates G_i, $i = 1, 2$. The ranking of the means and the symmetry assumption are sufficient to ensure that $k_i(p_j) > 0$, which in turn implies that the agent's problem is concave. A formal proof of this assertion is integrated into the proof of the upcoming Proposition 5.

denote the random variables that have distributions H_i and G_i, respectively.

Consider first matched components. Since two matched components each have symmetric densities and the same mean, the random variable described by their difference, $X_{H_1} - X_{H_2}$ or $X_{G_1} - X_{G_2}$, has mean zero and a symmetric density. Therefore, it is equally likely to be positive or negative. In other words, the two agents are equally likely to have the best performance, conditional on the components being matched. It is only if the components are mismatched that the two agents are not equally likely to win. Thus, any difference in the two agents' strategic considerations are driven by differences in what happens if the components are mismatched. In particular, the agent who views actions as strategic complements is whichever agent is less likely to win "on balance" when the components are mismatched, i.e. when taking the average over the two cases in which the components are mismatched.

Thus, consider the case of mismatched components, and assume that agent i draws from H_i and agent j from G_j. Conditional on this mismatch, the random variable described by the difference between agent i's performance and agent j's performance, $X_{H_i} - X_{G_j}$, has mean $\mu_H - \mu_G > 0$ and a symmetric density. Thus, it is positive with a probability in excess of $\frac{1}{2}$. In other words, agent i is more likely to win in this case, but less likely to win when he is the agent who draws from the bad component.

On average, agent 1's expected winning probability in the two mismatched cases is $\frac{1}{2}\left(\Pr(X_{H_1} - X_{G_2} > 0) + \Pr(X_{H_2} - X_{G_1} < 0)\right)$, or

$$\frac{1}{2}\left(\Pr(X_{H_1} - X_{G_2} > 0) + 1 - \Pr(X_{H_2} - X_{G_1} > 0)\right)$$

Thus, agent 1 is on balance less likely to win than agent 2 in the mismatched cases if he is the agent who enjoys less of an advantage in the best-case scenario where his good component is matched with his rival's bad component, or

$$\Pr(X_{H_1} - X_{G_2} > 0) < \Pr(X_{H_2} - X_{G_1} > 0). \tag{6}$$

Hence, determining whether agent 1 considers actions to be strategic complements boils down to checking this last inequality. If the inequality is replaced with equality, then there is no strategic interaction and equilibrium is in strictly dominant strategies.

To this end, the relative "spread" of the components are of key importance. Since they have the same median, H_1 and H_2 cross at μ_H. Say that H_i single-crosses

H_j from above if $H_i(x) > H_j(x)$ for all $x \in (-\infty, \mu_H)$ and $H_i(x) < H_j(x)$ for all $x \in (\mu_H, \infty)$, and extend this definition to G_i and G_j in the obvious way. Roughly speaking, H_i is flatter than H_j and in this sense more spread out, whereas H_j is more concentrated around its mean. Equivalently, the two distributions are ranked in terms of the peakedness order, which in turn is closely related to the dispersive order. See Shaked and Shanthikumar (2007, chapter 3) for details. Note that H_i is a mean-preserving spread over H_j. As an example, if H_i and H_j belong to the same location-scale family, then the single-crossing condition is satisfied if the scale parameter is larger for H_i than for H_j.

Now, adding the assumption that the densities are unimodal, the peakedness order is preserved under convolution (Shaked and Shanthikumar (2007, Theorem 3.D.4)). The implications is that if H_i single-crosses H_j from above and G_j single-crosses G_i from above, then the distribution of $X_{H_i} - X_{G_j}$ single-crosses the distribution of $X_{H_j} - X_{G_i}$ from above (the random variables $-X_{G_j}$ and $-X_{G_i}$ are ranked the same as X_{G_j} and X_{G_i} in terms of the peakedness order). Since these two random variables have the same positive mean, $\mu_H - \mu_G > 0$, the former is therefore less likely to be positive, implying that $\Pr(X_{H_i} - X_{G_j} > 0) < \Pr(X_{H_j} - X_{G_i} > 0)$. Intuitively, when H_i and G_j are matched, they are both so noisy that the winner is more likely to be determined by an outlier performance, which works to the advantage of the agent drawing from the bad component. On the other hand, when H_j and G_i are matched, the agent with the bad component is much less likely to win since the distributions are more concentrated around their respective means.

Proposition 4 *Assume that all components have symmetric densities and that $\mu_{H_i} = \mu_H$ and $\mu_{G_i} = \mu_G$, $i = 1, 2$. Then, agent 1 views actions as strategic complements if and only if (6) holds. A sufficient condition for (6) is that all densities are unimodal and that H_1 single-crosses H_2 from above and G_2 single-crosses G_1 from above.*

Since H_1 is a mean-preserving spread over H_2 and G_2 a mean-preserving spread over G_1, it follows that $\sigma^2_{H_1} > \sigma^2_{H_2}$ and $\sigma^2_{G_2} > \sigma^2_{G_1}$. Then, $\text{Var}[X_1|a_1]$ increases faster (or decreases more slowly) with a_1 than $\text{Var}[X_2|a_2]$ does with a_2 if the impact functions are the same. Indeed, if $\sigma^2_{H_1} > \sigma^2_{G_2} > \sigma^2_{G_1} > \sigma^2_{H_2}$ then it is possible that $\text{Var}[X_1|a_1]$ is globally increasing in a_1 and $\text{Var}[X_2|a_2]$ is globally decreasing in a_2. In this case, agent 1 becomes more error prone, and agent 2 less so, when effort increases. The following corollary illustrates Proposition 4 for two special cases that have already been alluded to.

Corollary 1 *If all components are normally distributed,* $X_{H_i} \sim \mathcal{N}(\mu_H, \sigma^2_{H_i})$ *and* $X_{G_i} \sim \mathcal{N}(\mu_G, \sigma^2_{G_i})$ *for* $i = 1, 2$*, then (6) holds if and only if* $\sigma^2_{H_1} + \sigma^2_{G_2} > \sigma^2_{H_2} + \sigma^2_{G_1}$*.*[7] *If the good and bad components belong to the same respective location-scale families, or* $H_i(x) = H_0(\frac{x-\mu_H}{\sigma_{H_i}})$ *and* $G_i(x) = G_0(\frac{x-\mu_G}{\sigma_{G_i}})$ *for* $i = 1, 2$*, with symmetric and unimodal densities, then a sufficient condition for (6) is that* $\sigma_{H_1} > \sigma_{H_2} > 0$ *and* $\sigma_{G_2} > \sigma_{G_1} > 0$*.*

Neither agent has globally stronger incentives. Thus, the identity of the agent who works hardest depends on preferences and abilities. One way to see this is that

$$\begin{aligned} k_i(0) &= \Pr(X_{H_i} - X_{G_j} > 0) - \Pr(X_{G_i} - X_{G_j} > 0) \\ &= \Pr(X_{H_i} - X_{G_j} > 0) - \frac{1}{2} = \frac{1}{2} - \Pr(X_{G_j} - X_{H_i} > 0) = k_j(1). \end{aligned}$$

Hence, $k_1(0) = k_2(1)$ and $k_1(1) = k_2(0)$. Since both functions are linear, it follows that $k_1(p) = k_2(1-p)$ and indeed $k_1(\frac{1}{2}) = k_2(\frac{1}{2})$.

Taking this a step further, it is always possible to construct homogeneous preferences and abilities such that $p_1(a_1) = p_2(a_2) = \frac{1}{2}$ in equilibrium.[8] Then, each of the four possible matches of components are equally likely. The two players are equally likely to win whenever the components are matched, but if agent 1 views actions as complements then he is on balance less likely to win if the components are mismatched. Hence, agent 1 is the underdog, even though he is the agent who views actions as complements. The argument extends when preferences and abilities are "weaker" such that best responses are lowered and $p_1(a_1) < \frac{1}{2}$ in equilibrium. In such cases, $p_2(a_2) > p_1(a_1)$ because $k_2(p_1)$ is higher than $k_1(p_2)$ when impacts are small. Thus, agent 2 works harder than agent 1, which adds to his advantage.

Proposition 5 *Consider a two-player contest in which all components have symmetric densities, with* $\mu_{H_i} = \mu_H$ *and* $\mu_{G_i} = \mu_G$*,* $i = 1, 2$*. Then,* $k_1(p) = k_2(1-p)$ *and* $k_1(\frac{1}{2}) = k_2(\frac{1}{2})$*. Hence, neither agent has globally stronger incentives than the other when equilibrium is not in strictly dominant strategies. If preferences and abilities are homogeneous, then the agent who views actions as strategic complements is the underdog whenever his impact is no larger than* $\frac{1}{2}$ *in equilibrium.*

[7] An implication is that equilibrium is in strictly dominant strategies if all components are normally distributed and $\sigma^2_{H_1} + \sigma^2_{G_2} = \sigma^2_{H_2} + \sigma^2_{G_1}$.

[8] By Proposition 6, $k_2(0) = k_1(1)$, or $\tau_0 = \kappa_0 + \kappa_1$, and as already mentioned $\tau_1 = -\kappa_1$. Using these in Example 1 proves that $p_1(a_1) = p_2(a_2) = \frac{1}{2}$ if $v_i = \frac{2}{2\kappa_0 + \kappa_1}$, $i = 1, 2$, in that set-up.

Appendix B considers a complementary specification of the components. At least one components has a monotonic density, which rules out symmetric densities. In exchange, it is possible to replace the single-crossing condition with a mean-preserving spread. The intuition for who views actions as strategic complements remains the same, but now there are cases in which one agent has globally stronger incentives.

4 Comparative statics: Preferences and abilities

This section begins by assuming that one agent becomes stronger in the traditional sense that his valuation increases. The formal analysis of the resulting comparative statics is entirely routine, since it is merely a matter of recording the consequences of shifting one of two monotonic best-response functions. Thus, the main point of this section is conceptual, specifically to reinforce the point that in the mixture model it is the agent's view of actions as strategic complements or substitutes that matter, rather than his status as favorite or underdog per se.

Imagine that v_2 increases. Since agent 2 now values the prize more highly, it is hardly surprising he works harder in equilibrium. The change in agent 1's equilibrium action is in turn determined by whether he views actions as complements or substitutes. In the former case, the increase in agent 2's action eggs agent 1 on, whereas it discourages him in the latter case.

Proposition 6 *Consider a two-player mixture contest in which v_2 increases. Then, a_2 weakly increases in equilibrium. On the other hand, a_1 weakly increases in equilibrium if $\kappa_1 > 0$ and weakly decreases in equilibrium if $\kappa_1 < 0$.*

Consider a contest designer who wishes to spur both agents to work harder. For instance, the action represents human capital accumulation, and the designer desires that everyone "levels up." In the mixture model, this can be achieved by given the agent who views actions as substitutes a bonus if he wins, such that his valuation of the prize increases. This makes him work harder, which in turn makes his competitor – who sees actions as complements – work harder as well. However, the wrinkle is that it may be the favorite who views actions as substitutes and the designer then ends up encouraging the agent who is already more likely to win. A similar kind of logic reappears in the next section, which considers extensions to sequential contests.

Incidentally, the conclusion in Proposition 6 extends to the case where agent 2's marginal costs decrease for all actions. However, a change in agent 2's impact function is more subtle, because this directly impacts agent 1's best response function as well.

5 Sequential contests and precommitment

Dixit (1987) considers a sequential contest with an exogenous order of moves, much as in a Stackelberg game. Information is complete and agents have the same valuations and cost functions. However, their impact functions may be different. Dixit asks whether the agent that moves first over- or undercommits effort relative to equilibrium effort in a simultaneous move game.

In contests with two agents, the basic strategic considerations are intuitively clear. Any agent is more likely to win the lower the competitor's effort is. In Dixit's words, this "makes it strategically desirable for [the leader] to precommit his effort level in such a way as to induce a lower effort from the [follower] in response. Whether this means a commitment at a higher or a lower level of one's own effort depends on whether the other's best response function has a negative or positive slope."

In the contest games that Dixit studies – which includes Tullock contests – the favorite views actions as strategic complements locally whereas the follower views them as strategic substitutes. Thus, if the favorite (underdog) moves first, then an increase (decrease) in his own action leads his competitor to respond in the desired way. However, in the mixture model, the favorite, i.e. the agent who is most likely to win the simultaneous move contest, may be the agent who views actions as strategic substitutes. In this sense, Dixit's result can be reversed in the mixture model.

Baik and Shogren (1992) extend Dixit's model by endogenizing the order of moves. In equilibrium, the underdog moves first and the favorite second. The logic of their argument extends to the mixture model, in the sense that it is the agent who views actions as substitutes that moves first. However, in the mixture model, this means that it may be the favorite that moves first. In equilibrium, both agents expend less effort than in the equilibrium of the simultaneous move game. Baik and Shogren (1992) observe that the social costs of effort is therefore lower than what is predicted by the simultaneous game.

6 Discussion

6.1 Incomplete information

It is easy to add independent private information about preferences and abilities to the model, but it does not add anything to this paper's research question. Since $F_j(x_j|a_j)$ is separable, the only part of agent j's strategy that agent i cares about is the expected value of player j's impact (taken over player j's types). This is sufficient to describe the ex ante expected distribution of agent j's performance and inform agent i's maximization problem. Consequently, all of agent i's types react in the same direction to an increase in agent j's expected impact. In this sense, all of agent i's types agree whether actions are strategic complements or substitutes and it is once again the mixture components that, through κ_1 and τ_1, determine which agent views actions as complements. A more precise discussion is in Kirkegaard (2022).

Since it is only the technologies that matter, Propositions 2-5 and Corollary 1 are effectively unchanged in the face of incomplete information about preferences and abilities. In Proposition 5, the last part holds whenever the expected impact of the agent who view actions as strategic complements is no larger than $\frac{1}{2}$ in equilibrium.

6.2 Difference-form contest success functions

Consider the special case in which technologies are homogeneous. Then, $k_i(p_j) = \kappa_0$ is a constant, i.e. it is independent of p_j, and it is the same for $i = 1, 2$. Indeed, the contest success function takes a particularly simple form in this case. First

$$\kappa_0 = \int G(x)h(x)dx - \int G(x)g(x)dx = \int G(x)h(x)dx - \frac{1}{2}.$$

This is at most $\frac{1}{2}$ since the first term on the far right is the expectation of G, which cannot be greater than one. Hence, $\kappa_0 \in (0, \frac{1}{2})$.

Second, integration by parts confirms that $t_i(p_j) = \frac{1}{2} - p_j\kappa_0$. Thus,

$$q_i(a_i, a_j) = \frac{1}{2} + (p_i(a_i) - p_j(a_j))\,\kappa_0. \tag{7}$$

Since $\kappa_0 \in (0, \frac{1}{2})$, this probability is interior, i.e. it is in $(0, 1)$. The agent with the higher equilibrium impact is more likely to win. This is simply because the distribution of his equilibrium performance first order stochastically dominates that of his competitor, given the mixture components are identity-independent by assumption.

The contest success function in (7) is a special version of a difference-form contest. The important feature is that the agent can neither guarantee that he wins nor that he loses. In contrast, Che and Gale's (2000) analysis of piece-wise linear difference-form contests is made complicated by the fact that winning probabilities of 0 or 1 can be achieved. This produces an interaction between agents' actions, and equilibrium is therefore not in dominant strategies.

As e.g. Konrad (2009) and Brown and Minor (2014) point out, in a rank-order tournament in which the difference between the two agent's error terms is a random variable that is uniformly and symmetrically distributed around zero, the contest success function (CSF) is also of the difference form, much as in (7). Brown and Minor (2014) utilize this to good effect in an elimination tournament, since equilibrium in the final stage is in dominant strategies.

Thus, the rank-order tournament can be used to microfound this type of difference-form CSF. However, it requires that the random variable in question is distributed in a very particular way. The mixture model presented in the current paper provides an alternative microfoundation. This does not require the mixture components to be distributed in any specific ways, but it does of course require performance to be determined by a mixture distribution. Moreover, although the two models give rise to the same CSF in contests with two agents, they diverge in contests with more agents.

By Proposition 2 and Corollary 1, there are contests with heterogeneous technologies in which equilibrium is in strictly dominant strategies. For instance, this occurs for power technologies whenever $\alpha_1\beta_1 = \alpha_2\beta_2$. In such cases, $k_1(p_2) = \kappa_0$ and $k_2(p_1) = \tau_0$ are constants. Then, the CSF is again of the difference-form, but generally speaking not symmetric. For instance, the counterpart to (7) for player 1 is

$$q_1(a_1, a_2) = \int G_2(x)g_1(x)dx + p_1(a_1)\kappa_0 - p_2(a_2)\tau_0.$$

Thus, it is also possible to microfound asymmetric difference-form CSFs.

Finally, when equilibrium is not in strictly dominant strategies, the CSF is no longer a difference-form CSF. The CSF now contains an additional term that depends on the product of $p_1(a_1)$ and $p_2(a_2)$.

6.3 Technological heterogeneity in other contests

As emphasized repeatedly already, the common assumption in the contest literature is that heterogeneity reflects either differences in valuations or in cost or impact functions. The underlying performance technology is typically held fixed. Indeed, in order to invoke any of the microfoundations for the Tullock CSF, the noise must be similarly distributed across agents. For instance, in Fullerton and McAfee's (1999) justification, the interpretation of the impact function is that it measures a number of "ideas", each of which is i.i.d. across agents. In other words, the quality of a random idea or trial is distributed the same across agents. Although there are exceptions, the Tullock CSF does not generally obtain if the distribution of ideas differ across agents. Nevertheless, allowing heterogeneity along this dimension seems economically to be no less relevant or interesting than heterogeneity along the more familiar dimensions.[9]

This subsection demonstrates that the microfoundations for the Tullock CSF can in fact accommodate certain limited kinds of technological heterogeneity. Starting with Fullerton and McAfee's (1999) microfoundation, agent i's performance is characterized by a distribution function of the form

$$F_i(x_i|a_i) = H(x_i)^{p_i(a_i)},$$

where H is a distribution function that is common to all agents and where $p_i(a_i) \geq 0$ is the agent's impact function. It is possible to slightly relax the assumption that H is identity-independent. Specifically, assume that $H_i(x_i) = H_0(x_i)^{\alpha_i}$, where H_0 is a parent distribution common to all agent, and where $\alpha_i > 0$ can vary from agent to agent. Thus, α_i reflects a kind of technological heterogeneity that is somewhat in the spirit of Section 3.1 in the sense that it speaks to the "strength" of the agent. Then,

$$F_i(x_i|a_i) = H_i(x_i)^{p_i(a_i)} = H_0(x_i)^{\alpha_i p_i(a_i)},$$

and the technological heterogeneity is isomorphic to a rescaling of the impact function in a setting with a common distribution function. In other words, technological heterogeneity in this context can essentially be duplicated by changing the impact

[9] For instance, consider a set of PhD candidates who are about to enter the academic job market. The job market paper of any given candidate can be thought of as the culmination of the work that arose from his best idea. While some candidates may have more ideas than others, it also seems true that some candidates simply have better ideas on average.

function. It does not add extra dimensions to player heterogeneity, unlike in the mixture model. Indeed, if $\alpha_i p_i(a_i) = \alpha_j p_j(a_j)$ then the performance of the two agents are identically distributed. Note that if the H_i's are different across each other in some arbitrary way, then generally speaking the CSF is no longer a Tullock CSF.

Hirschleifer and Riley's (1992) microfoundation involves multiplicative noise. In particular, the agent's performance is his impact multiplied by a noise term that is exponentially distributed with mean one. As a result, the agent's performance is exponentially distributed with mean $p_i(a_i)$. However, the mean of the multiplicative noise can be made to vary from agent to agent. If the mean of agent i's noise term is $\alpha_i > 0$, then the agent's performance is exponentially distributed with mean $\alpha_i p_i(a_i)$. Thus, this kind of technological heterogeneity is exactly the same as in the version of Fullerton and McAfee's model just mentioned. Jia's (2008) microfoundation also features multiplicative noise and the same observation applies to that model as well, as can be seen formally from Jia's (2008) Corollary 1.

Turning next to Lazear and Rosen's (1981) rank-order tournament, the standard assumption is that the additive noise is identically distributed across agents. However, this assumption can be relaxed, allowing the noise to have different "spread". See e.g. Imhof and Kräkel (2016). Technologies are then heterogeneous in the spirit of Section 3.2. However, contrary to Section 3.2, the variance is independent of the action in that model. Likewise, it can be shown that if the noise terms have identity-dependent but symmetric densities, then best-response functions are hump-shaped and the favorite views actions as strategic complements. This need not be the case if the densities are not symmetric.

7 Conclusion

This paper introduces the mixture model of contests. The mixture model makes it possible to study heterogenous technologies, i.e. situations in which agents are so fundamentally different that they cannot exactly replicate the distribution of the performance of their competitor.

While heterogenous technologies can be modeled in other ways, the mixture model provides a particularly stark illustration of the consequences of this kind of heterogeneity. In particular, in the two-player version of the model, the technologies completely "take over" in terms of determining the nature of the strategic considerations. That is, it is solely the technologies that decide who views actions as strategic complements or substitutes. Preferences, costs, or other similar characteristics are irrelevant. Thus, unlike in other contest models, it may be the underdog who views actions as strategic complements. When this is so, the comparative statics are evidently different from the more traditional contest models. In other words, technological heterogeneity can overturn some of the conventional wisdom in contest theory.

The mixture model is sufficiently tractable that it allows the study of many-player contests with incomplete information. A companion paper, Kirkegaard (2022) examines such contests, assuming homogenous technologies. That paper challenges the robustness of some of the insights derived from the more standard contest models.

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Appendix A: Omitted proofs

Proof of Proposition 1. The main body of the text contains proves the proposition, using as a critical step the property that $\sum_{i=1}^{2} \frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j} = 0$. A direct proof of this property follows from the fact that

$$\begin{aligned}
\frac{\sum_{i=1}^{2} \frac{\partial^2 q_i(a_i,a_j)}{\partial a_i \partial a_j}}{p_1'(a_1)p_2'(a_2)} &= \int_{\underline{x}}^{\overline{x}} \left(\left(H_2(x) - G_2(x)\right)\left(h_1(x) - g_1(x)\right) + \left(H_1(x) - G_1(x)\right)\left(h_2(x) - g_2(x)\right)\right) dx \\
&= \left[\left(H_1(x) - G_1(x)\right)\left(H_2(x) - G_2(x)\right)\right]_{\underline{x}}^{\overline{x}} \\
&= 0,
\end{aligned}$$

where $\underline{x}$ and $\overline{x}$ is the upper and lower end-point of the support, respectively. The proposition now follows. ■

Proof of Proposition 2. Since $k_1'(p_2) + k_2'(p_1) = 0$, it follows that $k_1'(p_2) > 0$ if $k_1'(p_2) - k_2'(p_1) > 0$. Since

$$k_i'(p_j) = \int \left(F^C(x|\alpha_j) - F^C(x|\beta_j)\right)\left(f^C(x|\alpha_i) - f^C(x|\beta_i)\right) dx$$

it holds that

$$\begin{aligned}
k_i'(p_j) &= \int \left(F^C(x|\beta_j) - F^C(x|\alpha_j)\right)\left(F^C(x|\beta_i) - F^C(x|\alpha_i)\right)\left(\frac{f^C(x|\beta_i) - f^C(x|\alpha_i)}{F^C(x|\beta_i) - F^C(x|\alpha_i)}\right) dx \\
&= \int \left(F^C(x|\beta_j) - F^C(x|\alpha_j)\right)\left(F^C(x|\beta_i) - F^C(x|\alpha_i)\right) \frac{\partial \ln\left(F^C(x|\beta_i) - F^C(x|\alpha_i)\right)}{\partial x} dx
\end{aligned}$$

and therefore

$$\begin{aligned}
k_1'(p_2) - k_2'(p_1) = \int & \left[\left(F^C(x|\beta_1) - F^C(x|\alpha_1)\right)\left(F^C(x|\beta_2) - F^C(x|\alpha_2)\right)\right] \\
& \times \frac{\partial}{\partial x} \ln\left(\frac{F^C(x|\beta_1) - F^C(x|\alpha_1)}{F^C(x|\beta_2) - F^C(x|\alpha_2)}\right) dx.
\end{aligned}$$

The term in square brackets is strictly positive for all interior x, by first order stochastic dominance. Utilizing the strict log-supermodularity of $-F_\eta^C$, the last term in the integrand is strictly positive when agent 1 has the stronger technology. Then, $k_1'(p_2) > 0$ and agent 1 considers actions to be strategic complements.

For the second part, integration by substitution confirms that

$$\begin{aligned}
t_i(p_j) &= \frac{\beta_i}{\beta_i+\beta_j} - \frac{\beta_i\left(\alpha_j-\beta_j\right)}{\left(\beta_i+\beta_j\right)\left(\alpha_j+\beta_i\right)}\overline{p}_j \\
k_i(p_j) &= \frac{\beta_j\left(\alpha_i-\beta_i\right)}{\left(\beta_i+\beta_j\right)\left(\alpha_i+\beta_j\right)} + \frac{\left(\alpha_i-\beta_i\right)\left(\alpha_j-\beta_j\right)\left(\alpha_i\beta_i-\alpha_j\beta_j\right)}{\left(\alpha_i+\alpha_j\right)\left(\beta_i+\beta_j\right)\left(\alpha_j+\beta_i\right)\left(\alpha_i+\beta_j\right)}\overline{p}_j
\end{aligned}$$

regardless of what the parent distribution F_0 is. Since

$$k_i'(p_j) = \frac{\left(\alpha_i-\beta_i\right)\left(\alpha_j-\beta_j\right)\left(\alpha_i\beta_i-\alpha_j\beta_j\right)}{\left(\alpha_i+\alpha_j\right)\left(\beta_i+\beta_j\right)\left(\alpha_j+\beta_i\right)\left(\alpha_i+\beta_j\right)},$$

the second part of the proposition follows. ■

Proof of Proposition 3. The objective is to prove that (5) holds. Since one of k_1 and k_2 is weakly increasing and the other weakly decreasing in its argument, it follows that $\min_{p\in[0,1]} k_1(p) \geq \max_{p\in[0,1]} k_2(p)$ if k_1 exceeds k_2 at both corners, i.e. when $p = 0$ and when $p = 1$. Let $\Delta(p) = k_1(p) - k_2(p)$, and recall that $\Delta'(p) = k_1'(p) - k_2'(p) = \kappa_1 - \tau_1$ is independent of p. Of course, $\Delta(p)$ and $\Delta'(p)$ depends on the parameters α_1, β_1, α_2, and β_2, but that dependence is suppressed from the notation. Given

$$\begin{aligned}
k_i(0) &= \int F^C(x|\beta_j)\left(f^C(x|\alpha_i) - f^C(x|\beta_i)\right)dx \\
k_i(1) &= \int F^C(x|\alpha_j)\left(f^C(x|\alpha_i) - f^C(x|\beta_i)\right)dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
\Delta(0) &= \int F^C(x|\beta_2)\left(f^C(x|\alpha_1) - f^C(x|\beta_1)\right)dx - \int F^C(x|\beta_1)\left(f^C(x|\alpha_2) - f^C(x|\beta_2)\right)dx \\
\Delta(1) &= \int F^C(x|\alpha_2)\left(f^C(x|\alpha_1) - f^C(x|\beta_1)\right)dx - \int F^C(x|\alpha_1)\left(f^C(x|\alpha_2) - f^C(x|\beta_2)\right)dx.
\end{aligned}$$

Recall that the region of interest is $\alpha_1 \geq \alpha_2 > \beta_2 \geq \beta_1$. At $(\beta_1, \alpha_1) = (\beta_2, \alpha_2)$, it holds by symmetry that $\Delta(0) = \Delta(1) = \Delta'(p) = 0$. Now increase α_1. Since this leads to a first-order stochastic improvement in agent 1's good mixture component, this causes $\Delta(0)$ to increase. Thus, $\Delta(0) > 0$ whenever $\alpha_1 > \alpha_2 > \beta_2 = \beta_1$. Still assuming $\beta_2 = \beta_1$, note that $\Delta(1) = \Delta(0) + \Delta'(p)$, with

$$\begin{aligned}
\frac{\partial \Delta'(p)}{\partial \alpha_1} &= \int \left(F^C(x|\alpha_2) - F^C(x|\beta_2)\right) f^C_\eta(x|\alpha_1)dx - \int F^C_\eta(x|\alpha_1)\left(f^C(x|\alpha_2) - f^C(x|\beta_2)\right) dx \\
&= \int \int_{\beta_2}^{\alpha_2} \left(F^C_\eta(x|z) f^C_\eta(x|\alpha_1) - F^C_\eta(x|\alpha_1) f^C_\eta(x|z)\right) dzdx \\
&= \int \int_{\beta_2}^{\alpha_2} F^C_\eta(x|z) F^C_\eta(x|\alpha_1) \left(\frac{f^C_\eta(x|\alpha_1)}{F^C_\eta(x|\alpha_1)} - \frac{f^C_\eta(x|z)}{F^C_\eta(x|z)}\right) dzdx \\
&> 0
\end{aligned}$$

for all $\alpha_1 \geq \alpha_2$, due to the strict log-supermodularity of $-F^C_\eta$. Since both $\Delta(0)$ and $\Delta'(p)$ strictly increases with α_1, it holds that $\Delta(0) > 0$ whenever $\alpha_1 > \alpha_2 > \beta_2 = \beta_1$.

Now starting from any point with $\alpha_1 \geq \alpha_2 > \beta_2 = \beta_1$, decrease β_1. Here, $\Delta(1)$ is decreasing in β_1, so the decrease in β_1 turns out to increase $\Delta(1)$ further. Hence, $\Delta(1) > 0$. Next,

$$\begin{aligned}
\frac{\partial \Delta'(p)}{\partial \beta_1} &= -\int \left(F^C(x|\alpha_2) - F^C(x|\beta_2)\right) f^C_\eta(x|\beta_1)dx + \int F^C_\eta(x|\beta_1)\left(f^C(x|\alpha_2) - f^C(x|\beta_2)\right) dx \\
&= -\int \int_{\beta_2}^{\alpha_2} \left(F^C_\eta(x|z) f^C_\eta(x|\beta_1) - F^C_\eta(x|\beta_1) f^C_\eta(x|z)\right) dzdx \\
&= -\int \int_{\beta_2}^{\alpha_2} F^C_\eta(x|z) F^C_\eta(x|\beta_1) \left(\frac{f^C_\eta(x|\beta_1)}{F^C_\eta(x|\beta_1)} - \frac{f^C_\eta(x|z)}{F^C_\eta(x|z)}\right) dzdx \\
&> 0 \text{ for all } \beta_1 \leq \beta_2.
\end{aligned}$$

Thus, as β_1 decreases, $\Delta(1)$ is positive and increases, while $\Delta'(p)$ decreases. This implies that $\Delta(0)$ increases. Since $\Delta(0)$ was positive to start, it follows that $\Delta(0) > 0$. In summary, $\Delta(0) > 0$ and $\Delta(1) > 0$ when $\alpha_1 \geq \alpha_2 > \beta_2 \geq \beta_1$ and either $\alpha_1 > \alpha_2$ or $\beta_2 > \beta_1$, or both. This completes the proof of the first part.

For power technologies,

$$k_i(0) = \frac{\beta_j\left(\alpha_i - \beta_i\right)}{\left(\beta_i + \beta_j\right)\left(\alpha_i + \beta_j\right)} \text{ and } k_i(1) = \frac{\alpha_j\left(\alpha_i - \beta_i\right)}{\left(\alpha_i + \alpha_j\right)\left(\alpha_j + \beta_i\right)},$$

and

$$\begin{aligned}\Delta(0) &= \frac{2\beta_1\beta_2(\alpha_1-\alpha_2)-(\alpha_1\alpha_2+\beta_1\beta_2)(\beta_1-\beta_2)}{(\alpha_1+\beta_2)(\alpha_2+\beta_1)(\beta_1+\beta_2)}\\ \Delta(1) &= \frac{(\alpha_1\alpha_2+\beta_1\beta_2)(\alpha_1-\alpha_2)-2\alpha_1\alpha_2(\beta_1-\beta_2)}{(\alpha_1+\beta_2)(\alpha_2+\beta_1)(\alpha_1+\alpha_2)}.\end{aligned}$$

Hence, if the two conditions in the proposition are satisfied, then $\Delta(0) \geq 0$ and $\Delta(1) \geq 0$. This proves the second statement in the proposition.

For the last statement, assume that the two conditions are satisfied. The task is then to verify that they remain satisfied as (i) α_1 increases or (ii) β_1 decreases. The last part is easy, because the right hand side of both inequalities are increasing in β_1. For the first part, note that if the first inequality in the proposition is satisfied, then

$$\beta_2-\beta_1 \geq \frac{(\alpha_2-\alpha_1)\,2\beta_1\beta_2}{\alpha_1\alpha_2+\beta_1\beta_2}$$

and

$$\frac{\partial}{\partial\alpha_1}\left(\alpha_1-\alpha_2-\frac{\alpha_1\alpha_2+\beta_1\beta_2}{2\beta_1\beta_2}(\beta_1-\beta_2)\right)=\frac{\alpha_2(\beta_2-\beta_1)+2\beta_1\beta_2}{2\beta_1\beta_2}\geq\frac{\alpha_2^2+\beta_1\beta_2}{\alpha_1\alpha_2+\beta_1\beta_2}>0,$$

which proves that the first inequality is still satisfied when α_1 increases. The proof that the second inequality remains satisfied when α_1 increases is similar. ■

Proof of Proposition 4. The intuition is explained in the main text. A formal proof follows. Consider

$$\begin{aligned}k_1'(p_2) &= \int (H_2(x)h_1(x)+G_2(x)g_1(x)-G_2(x)h_1(x)-H_2(x)g_1(x))\,dx\\ &= \Pr(X_{H_2}<X_{H_1})+\Pr(X_{G_2}<X_{G_1})-\Pr(X_{G_2}<X_{H_1})-\Pr(X_{H_2}<X_{G_1})\\ &= \Pr(X_{H_1}-X_{H_2}>0)+\Pr(X_{G_1}-X_{G_2}>0)-\Pr(X_{H_1}-X_{G_2}>0)-\Pr(X_{H_2}-X_{G_1}<0),\end{aligned}$$

but since the random variables $(X_{H_1}-X_{H_2})$ and $(X_{G_1}-X_{G_2})$ have densities that are symmetric around zero, it follows that $\Pr(X_{H_1}-X_{H_2}>0)=\Pr(X_{G_1}-X_{G_2}>0)=\frac{1}{2}$. Thus,

$$\begin{aligned}k_1'(p_2) &= 1-\Pr(X_{H_1}-X_{G_2}>0)-\Pr(X_{H_2}-X_{G_1}<0)\\ &= \Pr(X_{H_2}-X_{G_1}>0)-\Pr(X_{H_1}-X_{G_2}>0)\end{aligned}$$

This proves that $k_1'(p_2) > 0$ if and only if (6) holds. The second part of the proposition was proven in the text. ■

Proof of Corollary 1. Assume that all mixture components are normally distributed. Then, the random variable $X_{H_2} - X_{G_1}$ has mean $\mu_H - \mu_G$ and variance $\sigma^2_{H_2} + \sigma^2_{G_1}$ while the random variable $X_{H_1} - X_{G_2}$ has mean $\mu_H - \mu_G$ and variance $\sigma^2_{H_1} + \sigma^2_{G_2}$. Thus,

$$\begin{aligned} k_1'(p_2) &= \Pr(X_{H_2} - X_{G_1} > 0) - \Pr(X_{H_1} - X_{G_2} > 0) \\ &= \Pr(X_{H_1} - X_{G_2} < 0) - \Pr(X_{H_2} - X_{G_1} < 0) \\ &= \Phi\left(\frac{0 - (\mu_H - \mu_G)}{\sigma^2_{H_1} + \sigma^2_{G_2}}\right) - \Phi\left(\frac{0 - (\mu_H - \mu_G)}{\sigma^2_{H_2} + \sigma^2_{G_1}}\right) \\ &= \Phi\left(\frac{\mu_G - \mu_H}{\sigma^2_{H_1} + \sigma^2_{G_2}}\right) - \Phi\left(\frac{\mu_G - \mu_H}{\sigma^2_{H_2} + \sigma^2_{G_1}}\right). \end{aligned}$$

Since $\mu_G - \mu_H < 0$, $k_1'(p_2) > 0$ if and only if $\sigma^2_{H_1} + \sigma^2_{G_2} > \sigma^2_{H_2} + \sigma^2_{G_1}$. This proves the first part of the proposition. The second part follows from applying the second part of Proposition 4. ■

Proof of Proposition 5. To begin, for agent 1

$$\begin{aligned} k_1(0) &= \Pr(X_{H_1} - X_{G_2} > 0) - \Pr\left(X_{G_1} - X_{G_2} > 0\right) = \Pr(X_{H_1} - X_{G_2} > 0) - \frac{1}{2} \\ k_1(1) &= \Pr(X_{H_1} - X_{H_2} > 0) - \Pr(X_{H_2} - X_{G_1} < 0) = \frac{1}{2} - \Pr(X_{H_2} - X_{G_1} < 0). \end{aligned}$$

This is an opportune time to prove the claim in the main body of the text that $k_i(p_j) > 0$ even without assuming that H_i first order stochastically dominates G_i. Since $X_{G_2} - X_{H_1}$ is symmetric around $\mu_G - \mu_H < 0$, the probability that it is negative is between $\frac{1}{2}$ and 1. It follows that $k_1(0) \in (0, \frac{1}{2})$. A similar argument proves that $k_1(1) \in (0, \frac{1}{2})$. Since $k_1(p_2)$ is an increasing or decreasing function, it is always between $k_1(0)$ and $k_2(1)$. Hence, $k_1(p_2) \in (0, \frac{1}{2})$ for all $p_2 \in [0, 1]$. By symmetry, the same arguments holds for agent 2 as well.

Indeed, for agent 2,

$$\begin{aligned} k_2(0) &= \Pr(X_{H_2} - X_{G_1} > 0) - \frac{1}{2} = 1 - \Pr(X_{H_2} - X_{G_1} < 0) - \frac{1}{2} = k_1(1) \\ k_2(1) &= \frac{1}{2} - \Pr(X_{H_1} - X_{G_2} < 0) = \frac{1}{2} - (1 - \Pr(X_{H_1} - X_{G_2} > 0)) = k_1(0). \end{aligned}$$

Thus, $k_2(0) = k_1(1)$ and $k_2(1) = k_1(0)$. Moreover, $k_1(p_2)$ and $k_2(p_1)$ are linear functions. Combining these two facts yields

$$\begin{aligned} k_1(p) &= (1-p)\,k_1(0) + pk_1(1) \\ &= (1-p)\,k_2(1) + pk_2(0) \\ &= k_2(1-p). \end{aligned}$$

Thus, there are only two possibilities. The first possibility is that $k_1(p_2)$ and $k_2(p_1)$ are constants, in which case equilibrium is in strictly dominant strategies. The only other possibility is that $k_1(p)$ and $k_2(p)$ cross each other on $p \in [0,1]$, in which case neither agent has globally stronger incentives. Since $k_1(p) = k_2(1-p)$, it is immediate that $k_1(\frac{1}{2}) = k_2(\frac{1}{2})$.

Finally, assume without loss of generality that agent 1 views actions as strategic complements, or $k_1'(p_2) > 0$. Assume agents have homogenous preferences and abilities. Then, no equilibrium can feature $p_2 < p_1 \leq \frac{1}{2}$ because this implies $k_2(p_1) > k_1(p_2)$, which in turn means that agent 2's best response is weakly bigger than agent 1's best response type-for-type, but this yields the contradiction that $p_2 \geq p_1$. Hence, if $p_1 \leq \frac{1}{2}$ then $p_2 \geq p_1$. Simple algebra confirms, with some abuse of notation, that

$$\begin{aligned} q_1(a_1,a_2) - q_2(a_2,a_1) &= p_1(1-p_2)\left(2\Pr(X_{H_1} - X_{G_2} > 0) - 1\right) \\ &\quad - p_2\,(1-p_1)\left(2\Pr(X_{H_2} - X_{G_1} > 0) - 1\right) \\ &< (p_1 - p_2)\left(2\Pr(X_{H_2} - X_{G_1} > 0) - 1\right), \end{aligned}$$

where the argument a_i has been omitted from $p_i(a_i)$ for notational simplicity. The inequality follows from $2\Pr(X_{H_1} - X_{G_2} > 0) - 1 < 2\Pr(X_{H_2} - X_{G_1} > 0) - 1$, by (6). Then, since $\Pr(X_{H_2} - X_{G_1} > 0) > \frac{1}{2}$, it follows that $q_1(a_1,a_2) - q_2(a_2,a_1)$ if $p_1(a_1) \leq p_2(a_2)$. This completes the proof that agent 1 is the underdog if he views actions as strategic complements and $p_1(a_1) \leq \frac{1}{2}$ in equilibrium. ∎

Proof of Proposition 6. The increase in agent 2's valuation shifts his reaction function up or out but does not change agent 1's reaction function. This changes the intersection of the reaction functions. Given that the reaction functions are monotonic and slope in opposite directions, a_2 weakly increases in equilibrium. At the same time, a_1 weakly increases if is is agent 1's reaction function that has a positive slope ($\kappa_1 > 0$) and weakly decreases if it is his reaction function that has a negative slope ($\kappa_1 < 0$).

These are weak rather than strict changes, because it is possible that equilibrium initially involved an action on the boundary of the action set. ■

Appendix B: More properties of the mixture model

B.1 Comparative statics of heterogeneous power technologies

Lemma 1 *Consider a two-player contest with heterogenous power technologies. Holding fixed α_2 and β_2, $k_1(p_2)$ is strictly increasing in α_1 and strictly decreasing in β_1 for all $p_2 \in [0,1]$.*[10] *On the other hand, for any $p_1 \in (0,1)$, $k_2(p_1)$ is strictly decreasing (increasing) in α_1 if $\alpha_1^2 > \alpha_2\beta_2$ ($\alpha_1^2 < \alpha_2\beta_2$) and strictly increasing (decreasing) in β_1 if $\beta_1^2 < \alpha_2\beta_2$ ($\beta_1^2 > \alpha_2\beta_2$). Thus, if $\alpha_1^2 > \alpha_2\beta_2 > \beta_1^2$, then a marginal increase in α_1 or a marginal decrease in β_1 leads $k_2(p_1)$ to decrease for all $p_1 \in (0,1)$.*

Proof. Since $k_i(p_j)$ is a convex combination of $k_i(0)$ and $k_i(1)$, it must increase (decrease) for all $\bar{p}_j$ if it increases (decreases) at both endpoints $k_i(0)$ and $k_i(1)$. Simple differentiation confirms that $k_1(0)$ and $k_1(1)$ strictly increases in α_1 and strictly decreases in β_1, as desired. Hence, $k_1(p_2)$ strictly increases for all $p_2 \in [0,1]$ if α_1 increases or β_1 decreases. For $k_2(p_1)$, it holds that

$$
\begin{aligned}
\frac{\partial k_2(0)}{\partial \alpha_1} &= 0 \text{ and } \frac{\partial k_2(1)}{\partial \alpha_1} = \frac{(\alpha_2\beta_2 - \alpha_1^2)(\alpha_2 - \beta_2)}{(\alpha_1 + \alpha_2)^2 (\alpha_1 + \beta_2)^2} \\
\frac{\partial k_2(0)}{\partial \beta_1} &= \frac{\left(\alpha_2\beta_2 - \beta_1^2\right)(\alpha_2 - \beta_2)}{(\alpha_2 + \beta_1)^2 (\beta_1 + \beta_2)^2} \text{ and } \frac{\partial k_2(1)}{\partial \beta_1} = 0.
\end{aligned}
$$

Thus, $k_2(p_1)$ is strictly decreasing (increasing) in α_1 for all $p_1 \in (0,1]$ if $\alpha_2\beta_2 - \alpha_1^2 < 0$ ($\alpha_2\beta_2 - \alpha_1^2 > 0$). Similarly, $k_2(p_1)$ is strictly increasing (decreasing) in β_1 for all $p_1 \in [0,1)$ if $\alpha_2\beta_2 - \beta_1^2 > 0$ ($\alpha_2\beta_2 - \beta_1^2 < 0$). This proves the lemma. ■

The first part of the lemma is intuitive. Simply put, the incentive to take a higher action is greater when the agent's good component is very good or his bad component is very bad. On the other hand, on a large portion of the parameter space – or more precisely when $\alpha_1^2 > \alpha_2\beta_2 > \beta_1^2$ – $k_2(p_1)$ is increasing in β_1 but decreasing in α_2. In this case, agent 2's incentives become weaker as agent 1's mixture components diverge from each other, or $\beta_1 \to 0$ and $\alpha_1 \to \infty$. In the limit, agent 1's mixture components are degenerate and agent 2 wins if and only if agent 1 draws from his bad component. Since agent 2 cannot effect the probability of that event, his incentives disappear.

[10] A stronger statement can be proven. Specifically, if H_1 (G_1) improves in the sense of first-order stochastic dominance, then $k_1(p_2)$ increases (decreases). This statement does not require power technologies. However, power technologies enables predictions about how $k_2(p_1)$ depends on agent 1's mixture components.

The case where $\alpha_1^2 > \alpha_2\beta_2 > \beta_1^2$ is a natural starting point because it is satisfied when the two agents have homogeneous power technologies. Thus, it also holds when technologies are "almost" homogeneous. More generally, the parameter space can be divided into four regions, depending on (i) whether $\alpha_1\beta_1 > \alpha_2\beta_2$ or $\alpha_1\beta_1 < \alpha_2\beta_2$ hold and (ii) whether $\alpha_1^2 > \alpha_2\beta_2 > \beta_1^2$ does or does not apply. Thus, define the regions

$$
\begin{aligned}
A' &= \{(\alpha_1, \beta_1, \alpha_2, \beta_2)\,|\alpha_1 > \beta_1,\ \alpha_2 > \beta_2,\ \alpha_1\beta_1 > \alpha_2\beta_2 \text{ and } \alpha_1^2 > \alpha_2\beta_2 > \beta_1^2\} \\
A'' &= \{(\alpha_1, \beta_1, \alpha_2, \beta_2)\,|\alpha_1 > \beta_1,\ \alpha_2 > \beta_2,\ \alpha_1\beta_1 < \alpha_2\beta_2 \text{ and } \alpha_1^2 > \alpha_2\beta_2 > \beta_1^2\} \\
B &= \{(\alpha_1, \beta_1, \alpha_2, \beta_2)\,|\alpha_1 > \beta_1,\ \alpha_2 > \beta_2, \text{ and } \alpha_1^2 < \alpha_2\beta_2\} \\
C &= \{(\alpha_1, \beta_1, \alpha_2, \beta_2)\,|\alpha_1 > \beta_1,\ \alpha_2 > \beta_2, \text{ and } \beta_1^2 > \alpha_2\beta_2\}
\end{aligned}
$$

In Figure 1, regions A' and A'' are to the north-west of the point on the 45° line where $\alpha_1 = \beta_1 = \sqrt{\alpha_2\beta_2}$, i.e. where the 45° line is intersected by the $\alpha_1\beta_1 = \alpha_2\beta_2$ curve. Further, A' is above the latter curve, A'' below it. The region C is to the north-east of the point just mentioned, and B to the south-west of it, but in both cases above the 45° line.

Proposition 7 *Consider a two-player contest with heterogenous power technologies. Holding fixed α_2 and β_2, the equilibrium values of a_1 and a_2 changes with α_1 and β_1 as follows:*

1. *A marginal increase in α_1 causes a_1 to increase in region A'', a_2 to decrease in region A' and C, and a_2 to increase in region B.*

2. *A marginal increase in β_1 causes a_2 to increase in region A', and a_1 to decrease in region A'', B, and C.*

Proof. Consider regions A' and A''. Here, $\alpha_1^2 > \alpha_2\beta_2 > \beta_1^2$. By the previous lemma, $k_1(p_2)$ is strictly increasing in α_1 and strictly decreasing in β_1 for all $p_2 \in [0, 1]$, $k_2(p_1)$ is strictly decreasing in α_1 and strictly increasing in β_1 for all $p_1 \in (0, 1)$. If α_1 increases, the increase in $k_1(p_2)$ and decrease in $k_2(p_1)$ in turn implies that agent 1's best response increases and agent 2's best response decreases. Thus, the fixed point changes as well. From Proposition 2, if $\alpha_1\beta_1 > \alpha_2\beta_2$ then $k_1'(p_2) > 0$ and therefore agent 1's best response is increasing in a_2, while $k_2'(p_1) < 0$ and therefore agent 2's best response is decreasing in a_1. Given how the reaction functions shift, it follows that a_2 decreases if a decrease is feasible, i.e. if $a_2 > \min A_2$ (the change in a_1 is ambiguous). By a similar argument a_1 increases if $a_1 < \max A_1$ and $\alpha_1\beta_1 < \alpha_2\beta_2$ (in this case, the change in a_2 is ambiguous). If β_1 increases, then the argument is reversed.

For regions B and C, the previous lemma once again makes it possible to infer how the best response functions change. The rest of the argument then follow the same logic as above. ∎

B.2 Monotonic densities and mean-preserving spreads

The next result complements the specification in Section 3.2 by imposing other regularity conditions on the mixture components. The result assumes, as in the main model, that H_i first order stochastically dominates G_i, $i = 1, 2$.

Proposition 8 *Consider a two-player contest in which $H_2 - G_2$ is weakly convex. Assume that either (i) $G_1 = G_2 = G$ is weakly convex and H_1 is a mean-preserving spread over H_2 or (ii) $H_1 = H_2 = H$ is weakly concave and G_1 is a mean-preserving contraction over G_2. In either case, agent 1 (weakly) views actions as strategic complements. Agent 1 has globally stronger incentives in case (i) but agent 2 has globally stronger incentives in case (ii).*

Proof. Assume that $H_2 - G_2$ is weakly convex. As a preliminary result, it will be shown that if agent 1 views actions as strategic complements and H_1 undergoes a mean-preserving spread or G_1 a mean-preserving contraction, then it remains the case that agent 1 sees actions as strategic complements. First,

$$k_1'(p_2) = \int (H_2(x) - G_2(x)) (h_1(x) - g_1(x)) \, dx$$

is assumed to be positive before agent 1's technology changes. If $H_2 - G_2$ is weakly convex and H_1 undergoes a mean-preserving spread then

$$\int (H_2(x) - G_2(x)) \, h_1(x) dx$$

weakly increases and it follows that agent 1 still views actions as strategic complements. Similarly, if G_1 undergoes a mean-preserving contraction then

$$\int (H_2(x) - G_2(x)) \, g_1(x) dx$$

weakly decreases and the result follows again.

Now apply this preliminary result to case (i) in the proposition. It is assumed that $G_1 = G_2 = G$. If it was the case that $H_1 = H_2$, then the two agents would have homogenous technologies and $k_1'(p_2) = 0$. However, since H_1 is a mean-preserving spread over H_2, the preliminary result implies that $k_1'(p_2) \geq 0$. A similar argument holds for case (ii). This proves the first statement in the proposition.

For the second part of the proposition, given that $k_1'(p_2) \geq 0$ and $k_2'(p_1) \leq 0$, (5) is satisfied if and only if $k_1(0) - k_2(0) \geq 0$. Now consider case (i) in the proposition and note that

$$\begin{aligned} k_1(0) - k_2(0) &= \int G_2(x)\,(h_1(x) - g_1(x))\,dx - \int G_1(x)\,(h_2(x) - g_2(x))\,dx \\ &= \int G(x)\,(h_1(x) - h_2(x))\,dx \end{aligned}$$

since $G_1 = G_2 = G$. Given that G is weakly convex and that H_1 is a mean-preserving spread over H_2, it follows that $k_1(0) - k_2(0) \geq 0$. This proves that agent 1 has globally stronger incentives in case (i).

By a similar argument, it sufficient to show that $k_2(1) - k_1(1) \geq 0$ in order to establish that agent 2 has globally stronger incentives, given that it is still the case that agent 1 views actions as strategic complements. In case (ii),

$$\begin{aligned} k_2(1) - k_1(1) &= \int H_1(x)\,(h_2(x) - g_2(x))\,dx - \int H_2(x)\,(h_1(x) - g_1(x))\,dx \\ &= \int H(x)\,(g_1(x) - g_2(x))\,dx \end{aligned}$$

since $H_1 = H_2 = H$. Given that H is weakly concave and G_1 is a mean-preserving contraction over G_2, it follows that $k_2(1) - k_1(1) \geq 0$. This completes the proof. ■

Consider case (i) in Proposition 8, where agent 1 has globally stronger incentives, and assume that the inequality in (5) is strict. Then, $k_1(0) > k_2(1)$. Even assuming that the two agents have homogenous preferences and abilities, it is always possible to find impact and cost functions that imply that they are so sensitive to incentives (the levels of k_1 and k_2) that equilibrium is at an action profile for which $p_1(a_1) = 1$ and $p_2(a_2) = 0$. Then, agent 1's draw is from H_1 and agent 2's draw from G_2. Since H_1 first order stochastically dominates $G_2 = G_1$, it follows that agent 1 is the favorite.

Conversely, in case (ii) in Proposition 8, impact and cost functions can be found for which $p_1(a_1) = 0$ and $p_2(a_2) = 1$ in equilibrium. Then, agent 2 is the favorite because $H_2 = H_1$ first order stochastically dominates G_1.

In both cases, it is agent 1 that views actions to be strategic complements. However, in one case he is the favorite and in the other case he is the underdog.