Contest Design with Stochastic Performance*

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Abstract

This paper studies optimal contest design in contests with noisy performance. Here, contest design is a team moral hazard problem that endogenizes the assignment rule that maps performance profiles into winning probabilities. The optimal design has similar features for a wide range of objective functions. It endogenizes standards for eligibility and the number of prizes that are awarded may be stochastic ex ante. The model sheds new light on preferential treatment in numerous settings, including guaranteed admission policies for select groups and heterogenous admissions standards. Finally, the approach derives endogenous, microfounded, and fully optimal contest success functions.

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1 Introduction

A broad range of economic interactions are contest-like in nature. For the purposes of this paper, think of a contest as an environment in which rival agents take costly actions that influence the probability with which one of a number of fixed and identical prizes is won. Examples include rent-seeking, lobbying, college admission, innovation contests, promotion contests, sports, etc. The winners need not necessarily be the agents who took the most costly actions. In fact, actions are not even directly observable in most contests. Instead, prizes are typically awarded on the basis of some noisy signal, often interpretable as performance.

In some settings it is natural to assume that prizes must be awarded to the agents with the "best" signal or performance. There are several design issues in such unbiased contests, many of which have been explored elsewhere.[1] This paper in contrast considers the diametrically opposite case in which there is no obligation to award the prizes to the agents with the best signals. Thus, the paper concentrates on the optimal design of biased contests. Depending on the application, this may be implemented as preferential treatment, affirmative action, nepotism, or the like.

A general and unifying model of contests is examined. Special cases have been explored before. For instance, it is known that the model can deliver microfoundations for the popular lottery contest success function (CSF) when the contest is *unbiased* and there is just one prize. However, it will be argued in the course of this paper that extensions to *biased* contests have not always stayed true to the premise of these microfoundations. The resulting analysis can be criticized as being ad hoc or poorly founded. This paper instead provides an internally consistent treatment of the optimal design of contests. An advantage of the approach is that contest design is based directly on the observables, i.e. the performance profile. In contrast, the lottery CSF approach relies on manipulating a black box that takes unobservable actions as inputs and it is therefore unclear how to translate the ensuing findings into practice. The paper thus has

[1] Design issues include questions concerning what the optimal set of contestants is and how they are selected, entry fees, number and distribution of prizes, etc. For these and related questions, see e.g. Taylor (1995), Fullerton and McAfee (1999), Moldovanu and Sela (2001), Che and Gale (2003), Drugov and Ryvkin (2020), and Fang, Noe, and Strack (2020).

both methodological and practical implications.

Consider a college admissions problem. Here, the student's high school GPA is observed. It is not uncommon in the US to differentiate between in-state and out-of-state students. For instance, in the UC system, an applicant from California who graduates in the top 9% of his high school cohort is guaranteed admission into one of the UC campuses. This guarantee is not extended to out-of-state students and the minimum entry requirements are likewise more stringent for out-of-state students.[2] Similarly, many countries distinguish between domestic and foreign workers on the labor market. For instance, Canadian universities are obligated to hire a Canadian applicant if one exists that meets the bar. Contrary to the guaranteed admission policy, however, such an applicant is not guaranteed a job since he may lose out to another Canadian applicant. A foreign applicant can be hired only if no Canadian candidate meets the bar.

It is natural to assume that the governments who mandate these policies care more about in-state or domestic individuals and that they explicitly seek to extend an advantage to these individuals ex post. However, the model examined in this paper can justify closely related policies even if the contest designer is primarily interested in incentivizing effort. That is, the focus is on contests as ex ante incentive schemes.

In these examples, agents must meet certain minimum requirements to have a chance of winning a prize. These "standards for eligibility" are part of the design, and are thus endogenous. Existing models typically abstract away from this design element but the current model is well suited to exploring the issue. This is pertinent in many kinds of contests and is relevant even if agents are symmetric ex ante. Consider research contests. The Ansari X Prize required a crewed spacecraft to be used twice in two weeks to enter space, a requirement that is clearly endogenous. In some contests, the requirements are very stringent indeed, and the prize may or may not be awarded as a result. Che and Gale (2003) mention the longitude rewards offered by the British government in 1714. These were predated by rewards offered by Spain and the Netherlands that were never awarded. The locomotive engine contest described by Che and Gale (2003)

[2] Compare the in-state and out-of-state instructions (accessed October 6, 2020) at https://admission.universityofcalifornia.edu/admission-requirements/freshman-requirements/.

drew ten entrants, of which only five were able to compete, and only one was able to complete the trial successfully. Similarly, the number of recipients of the National Medal of Science and the MacArthur Fellowship, or the "genius grant," varies greatly from year to year. Ex ante, the number of prizes that are awarded appears to be stochastic.[3,4] This is consistent with the model's predictions.

The following simple model is proposed. First, actions are not directly observable but a noisy and observable signal is produced by each agent. Typically, the noisy signal, q_i, can be thought of as the stochastic quality of agent i's performance. In a promotion contest among salespeople, a salesman's performance is his volume of sales. In innovation contests, a firm's performance is the quality of its innovation. In a competition for a merit-based scholarship or a seat at college, a student's performance includes his GPA to date. Similarly, a lobbyist's performance is how compelling he can make his agenda or proposal sound. The agent's action, a_i, impacts the distribution, $G_i(q_i|a_i)$, of his performance. It is unclear what the most reasonable specification of $G_i(q_i|a_i)$ is, and in any case it is probably sensitive to the application.[5]

The stochastic performance model nests popular CSFs. First, all-pay auctions or deterministic contests trivially arise if $G_i(q_i|a_i)$ is degenerate such that performance and action coincide. Second, in Lazear and Rosen's (1981) rank-order tournament, the action shifts the location of the non-degenerate distribution function. Finally, there are yet other specifications of $G_i(q_i|a_i)$ for which the probability that agent i delivers the best performance exactly reduces to the lottery CSF. Fullerton and McAfee's (1999) research tournament with a single prize is one such example.[6] However, if this is how the lottery CSF is justified,

[3] See https://www.nsf.gov/od/nms/medal.jsp and https://www.macfound.org/. By law, up to 20 National Medals of Science may be awarded yearly but the number averaged just under 10/year from 1962 to 2014. Between 20 and 30 MacArthur Fellowships are awarded each year.

[4] In some promotion contests, it may happen that no agent is promoted if they all perform terribly. In hiring, an extra hire may sometimes be made if two exceptional candidates apply.

[5] Recently, Bastani, Giebe, and Gürtler (2019) have independently proposed a virtually identical model in the single-prize case. However, their focus is on comparative statics in unbiased contests. See also Ryvkin and Drugov (2020) and Drugov and Ryvkin (2020).

[6] For other justifications in this vein, see Hirschleifer and Riley (1992), Clark and Riis (1996), Baye and Hoppe (2003), and Jia (2008). Skapardas (1996) and Clark and Riis (1998) instead take an axiomatic approach to justifying the lottery CSF. Corchón and Dahm (2011) consider a designer who cannot commit but who is not an expected utility maximizer. These microfoundations are influential and emphasized in e.g. the surveys by Konrad (2009), Vojnonić

internal consistency demands that any extension that moves beyond unbiased contests must continue to respect the basic stochastic performance premise.

Thus, the idea is to view the problem as a kind of contracting or team moral hazard problem, with the distributions $G_i(q_i|a_i)$ as the primitives. Instead of offering wage schedules as in Holmström (1982), it is winning probabilities that are manipulated to incentivize effort. The task is to design and commit to an "assignment rule" that maps performance profiles into winning probabilities subject to incentive compatibility constraints. This is a well-defined and entirely unambiguous problem. Hence, there is no reason a priori to impose ad hoc assumptions on the functional form that the biased CSF must take or to restrict attention to certain nice $G_i(q_i|a_i)$. In sum, the contract theory approach makes it possible to handle stochastic performance in much more generality. Note, however, that adverse selection is not considered.

The general design principle turns out to be the same for a wide range of objective functions. The optimal assignment rule is a deterministic function of performance profiles. It can be implemented by assigning to each agent a score that is a compromise between the preferences of the designer and the necessity to provide incentives. The agent with the highest overall score wins. Thus, the paper identifies a guiding principle for contest design.

Agents' likelihood-ratios play a key role in providing incentives. This is consistent with insights from the standard principal-agent model where the likelihood-ratio can be thought of as the incentive weight of any give wage. Negative likelihood-ratios should be punished, which is where rationing comes in. This is implemented by imposing endogenous standards for eligibility that may or may not be met in equilibrium. Similarly, very large likelihood-ratios should be rewarded if at all possible, which is where policies like guaranteed admission and preferential hiring enter the picture.

The way in which contest design is approached has significant methodological, conceptual, and practical implications. A popular approach to contest design is based on directly manipulating the CSF. This is problematic as the CSF is typically not a primitive of the model. Rather, it is just a reduced form that integrates

(2015), Corchón and Serena (2018), and Fu and Wu (2019). For other surveys on biased contest design, see Mealem and Nitzan (2016) and Chowdhury, Esteve-González, and Mukherjee (2019).

out the uncertainty over signals to calculate the agent's winning probability as a function of the action profile. A common version of this approach is to start with the unbiased lottery CSF – where the (unobserved) action mathematically turns out to play the same role as an (unobserved) number of tickets in an imaginary lottery – and then either gift or tax the agent to effectively alter the number of lottery tickets that he has. Methodologically, however, it is unclear how to carry out this taxation or transfer if the number of lottery tickets is unobservable to begin with.[7] Stated differently, while the unbiased lottery CSF has been microfounded, no attempt has been made to microfound the biased lottery CSF. Indeed, in Fullerton and McAfee's (1999) model, the endogenous CSF that is implied by the contest design proposed here is not a lottery CSF at all.

It is also worth emphasizing how hard it is to draw practical implications from the biased lottery approach. For instance, how should one go about levelling the playing field by ensuring that each agent has an equal number of lottery tickets when this number is not observable? In other words, it is hard to see how policy recommendations should be implemented in practice since all results and predictions relate to something that is unobservable. The approach in the current paper instead allows the observable variables to take centre stage since the assignment rule is explicitly based on the observed performance profile.

The existing literature based on manipulating the lottery CSF claims that the optimal design in a one-prize, two-agent contest implements a completely level playing field for almost all interesting objective functions. Stated differently, the two agents are equally likely to win the contest in equilibrium. However, this conclusion does not hold in the stochastic performance model. In fact, which agent should be favored depends on the underlying distribution of noise and on the designer's objective function. When the objective is to maximize total effort, a simple example shows that the optimal design can produce an improvement of up to 47% over the existing approach. It is also the case that the literature has underestimated the value to the designer of being able to ration or withhold the prize. Rationing, when possible, is generally speaking optimal.

[7] In contrast, if actions are observable then it appears unlikely that a lottery would be used in the first place. The all-pay auction is conceptually more satisfying in this regard as the premise is exactly that the action (or bid) is perfectly observable.

2 Contests with stochastic performance

This section lays out the basic model and formulates the design problem.

2.1 Contest primitives

There is a fixed set $N = \{1, ..., n\}$ of risk neutral contestants or agents. Agent i takes costly action $a_i \in \mathbb{R}_+$. Costs are normalized to be linear in a_i. Since costs are increasing in a_i, the action can often be interpreted as effort. The action influences the distribution of the agent's signal or performance, q_i. The distribution function is written $G_i(q_i|a_i)$. This is atomless whenever $a_i > 0$, in which case it has density $g_i(q_i|a_i) > 0$ and support $[\underline{q}_i, \overline{q}_i]$, which may or may not be bounded above or below. Note that the support is the same for all strictly positive actions. If $a_i = 0$, the possibility that the distribution is degenerate at $q_i = \underline{q}_i$ is allowed. Given actions, agents' signals are statistically independent.

The designer can award up to m identical and indivisible prizes, with $n > m \geq 1$. For now, think of m as well as the size or nature of each prize as exogenous; either can be endogenized in a second step. For simplicity and to facilitate comparison with the standard contest literature, much of the later part of the paper in fact focuses on the case with a single prize.

Agent i assigns some exogenous value $v_i > 0$ to winning a prize. Each agent can win at most one prize. The value of losing is zero. The outside option is likewise worth zero. Thus, the participation constraint is trivially satisfied because $a_i = 0$ guarantees a non-negative payoff. There is no entry fee, for now.

Special cases of the model have been considered before. For example, an all-pay auction is a contest with no noise. Here, $G_i(q_i|a_i)$ is degenerate for all actions, such that $q_i = a_i$ with probability one. Then, the agents with the highest actions win in an unbiased contest. In this paper, such distributions are ruled out. In Lazear and Rosen's (1981) rank-order tournament, the noise is derived from an agent who can shift the location of the distribution. Formally, $q_i = f_i(a_i) + \varepsilon_i$, where ε_i is the realization of a random variable. Finally, different specifications have been used in the literature to provide microfoundations for the lottery contest success function. This includes a model by Fullerton and McAfee (1999). The lottery CSF is revisited in Section 5.

2.2 Assignment rules and the moral hazard problem

Let Ω denote the collection of all permissible sets of winners, and let ω denote an element of Ω. Thus, ω describes an assignment of prizes. It is possible that there are restrictions on Ω, which may reflect the inability of the designer to commit to certain assignments or legal roadblocks that make certain assignments impossible to implement. For instance, if the designer cannot commit to ration prizes then any $\omega \in \Omega$ is restricted to have m distinct members.

Rationing is said to be possible if, for any $\omega \in \Omega$, it holds that if some agent i is a member of ω, or $i \in \omega$, then it is also the case that $\omega \backslash \{i\} \in \Omega$. That is, it is possible to withhold one or more prizes and ω is thus not restricted to have m members.[8] No prize is awarded if $\omega = \emptyset$. A legal restriction that a certain percentage of prizes must be awarded to women or minorities would likewise imply that not all constellations of winners are feasible. Thus, Ω is taken to be exogenous throughout.

To rule out trivialities, it is assumed that for any agent i there exists some $\omega \in \Omega$ for which $i \in \omega$ and some $\omega' \in \Omega$ for which $i \notin \omega'$. In other words, there is at least one feasible assignment that entails agent i winning a prize but he is not guaranteed to win a prize.

A contest elicits effort from agents. Hence, designing a contest is at heart a moral hazard problem. Thus, familiar logic can be applied. First, it is assumed that the entire performance profile is observed. A biased contest is then one in which the winners are not necessarily the agents with the highest performance.

Thus, let $P_\omega(\mathbf{q})$ denote the probability that the group $\omega \in \Omega$ wins, given the performance profile $\mathbf{q} = (q_1, q_2, ..., q_n)$. Let $\mathbf{P} = \{P_\omega\}_{\omega \in \Omega}$ denote the ensuing "assignment rule." This is the endogenous design instrument. It is explicitly assumed that the designer can credibly and fully commit to any feasible assignment rule; any and all restrictions are built into Ω. The feasibility constraints are that $P_\omega(\mathbf{q}) \in [0, 1]$ for all $\omega \in \Omega$ and that $\sum_{\omega \in \Omega} P_\omega(\mathbf{q}) = 1$, for all $\mathbf{q}$. Since $P_\omega(\mathbf{q}) = 0$ is permitted for any given ω, there is more flexibility in being able to manipulate $\mathbf{P}$ than being able to restrict Ω.

While it is easiest to think of prizes as being identical and indivisible, the

[8] This is the most flexible kind of rationing. In some contests, only partial rationing may be possible. For instance, it may be impossible to withhold more than half the prizes.

model does permit another interpretation. Let $m = 1$ denote the size of a perfectly divisible prize and let $\Omega = N \cup \{0\}$. Then, $P_{\{i\}}(\mathbf{q})$ can be interpreted as the share of the prize that agent i receives, with $P_{\{0\}}(\mathbf{q})$ being the share that the designer retains for herself. For instance, $m = 1$ may be the total wage budget, in which case $P_{\{i\}}(\mathbf{q}) \geq 0$ can be thought of as a limited liability constraint.

Agents care only about winning or losing but not about the outcomes for other agents. With some abuse of notation, the probability that agent i wins a prize, given $\mathbf{q}$, is

$$P_i(\mathbf{q}) = \sum\nolimits_{\{\omega \in \Omega | i \in \omega\}} P_\omega(\mathbf{q}). \tag{1}$$

It is often convenient to write $P_i(\mathbf{q})$ as $P_i(q_i, \mathbf{q}_{-i})$, where $\mathbf{q}_{-i}$ denotes the performance profile of agent i's rivals. Note that different assignment rules may yield the same reduced winning probability $P_i(\mathbf{q})$ when there are multiple prizes.

Let $\mathbf{a}_{-i}$ denote the vector of actions by agent i's rivals. Given $\mathbf{a}_{-i}$, agent i's expected utility from action a_i is now

$$U_i(a_i, \mathbf{a}_{-i}|\mathbf{P}) = v_i \int \left(\int P_i(q_i, \mathbf{q}_{-i}) g_i(q_i|a_i) dq_i \right) \prod_{j \neq i} g_j(q_j|a_j) d\mathbf{q}_{-i} - a_i, \tag{2}$$

since signals are statistically independent. The factor after v_i integrates out the uncertainty over performance profiles and thus expresses agent i's ex ante winning probability as a function only of the action profile. In the language of contest theory, this is the CSF. Evidently, the CSF is endogenous. The important point is that it is endogenized by manipulating the assignment rule, which in turn is based on something observable. Thus, the CSF is not a black box.

For an action profile $\mathbf{a} = (a_1, a_2, ..., a_n)$ to be implementable, it must constitute a Nash equilibrium of the contest game. The equilibrium can be manipulated by making changes to the assignment rule. Attention is restricted to pure strategy implementation throughout.

2.3 The contest environment

Unless explicitly mentioned, no functional form is imposed on the distributions $G_i(q_i|a_i)$. Thus, the aim is to analyze contests in some generality. However, there are technical road blocks. Thus, the analysis relies on the standard first-order approach known from classic moral hazard problems. The validity of this approach places technical restrictions on $G_i(q_i|a_i)$.

It will be assumed from now on that actions are continuous and that $g_i(q_i|a_i)$ is differentiable with respect to a_i when $a_i > 0$. Agent i's likelihood-ratio,

$$L_i(q_i|a_i) = \frac{1}{g_i(q_i|a_i)} \frac{\partial g_i(q_i|a_i)}{\partial a_i},$$

plays an important role. A common assumption is that $L_i(q_i|a_i)$ is weakly increasing in q_i. For expositional simplicity, this paper assumes that $L_i(q_i|a_i)$ is strictly increasing in q_i. This is the *monotone likelihood-ratio property* (MLRP).

Assumption (MLRP): $L_i(q_i|a_i)$ is strictly increasing in q_i for all $a_i \in \mathbb{R}_{++}$ and all $i \in N$.

The MLRP implies that a higher action makes a lower performance less likely. Thus, agent i's expected performance, $\mathbb{E}[q_i|a_i]$, is strictly increasing in a_i.

In the standard contracting literature, the role of the MLRP is to ensure that wage schedules are monotonic in signals. It plays a similar role here as the standard technique can be applied whenever $P_i(q_i, \mathbf{q}_{-i})$ is non-decreasing in q_i in equilibrium. Specifically, Rogerson (1985) combines the MLRP with a *convexity of the distribution function condition* (CDFC) that assumes that $G_i(q_i|a_i)$ is convex in a_i for all q_i. The CDFC implies that the term in the parenthesis in (2) is concave in a_i for any monotonic $P_i(\mathbf{q})$. The easiest way to see this is by using integration by parts. Thus, $U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})$ is concave in a_i, given $\mathbf{a}_{-i}$. Consequently, the first-order condition identifies a best response.

Assumption (CDFC): $G_i(q_i|a_i)$ is convex in a_i for all $q_i \in [\underline{q}_i, \overline{q}_i]$ and all $i \in N$.

The Fullerton and McAfee (1999) model satisfies the MRLP and the CDFC. Indeed, Rogerson's (1985) leading example is a special case of that model. Thus, this paper concentrates on contests where the MLRP and the CDFC hold.

3 Assignment independent contests

This section begins the analysis by solving a tractable yet flexible class of contests. The next section considers a more general contest environment.

3.1 The objective function and the design problem

The designer is endowed with a Bernoulli utility function, $\pi(\mathbf{q}, \mathbf{a})$, which is allowed to depend on the performance profile and the action profile. The main restriction is that the designer does not care about the identity of the winners or the assignment. In other words, her preferences are *assignment independent.*

The designer may care directly about the action profile. The most common assumption in the contest literature is that the designer is interested in maximizing total effort, or $\pi(\mathbf{q}, \mathbf{a}) = \sum_{j \in N} a_j$. Another common objective function in single-prize contests involves maximizing the highest action, or $\pi(\mathbf{q}, \mathbf{a}) = \max\{a_1, a_2, ..., a_n\}$. In these contests, q_i is just some signal about effort but q_i is in itself of no interest to the designer. For instance, a_i may capture human-capital accumulation that is of importance in the long run, whereas q_i is performance in the short run that is of lesser or no value but is more readily observable.

However, the designer may also care directly about performance. For instance, consider a contest for one or several promotions among salesmen akin to Lazear and Rosen (1981). Here, the employer is presumably not directly interested in the salesmen's efforts but rather in the total volume of sales, or $\pi(\mathbf{q}, \mathbf{a}) = \sum_{j \in N} q_j$. Alternatively, consider $\pi(\mathbf{q}, \mathbf{a}) = \max\{q_1, q_2, .., q_n\}$. This Bernoulli utility function applies when the designer only cares about the best performance, even though this may or may not equal the winner's performance. For instance, a firm may pursue the best product design proposed by a disparate group of in-house developers, yet may at the same time choose to promote a developer whose own design was inferior to handle the product launch.

Since the designer's preferences are assignment independent, she is ex post indifferent to how prizes are assigned. Note that the commitment problem is less severe in this case, as there is no ex post incentive to deviate from the promised assignment rule. Moreover, the designer's expected utility is only a function of the induced action profile. It does not depend on the assignment rule used to achieve the action profile. Thus, the designer sets out to design the assignment rule $\mathbf{P}$ to implement an action profile $\mathbf{a}$ that maximizes her expected utility,

$$U_0(\mathbf{a}) = \mathbb{E}[\pi(\mathbf{q}, \mathbf{a})|\mathbf{a}].$$

Note that the CDFC implies that $\mathbb{E}[\sum_{j\in N} q_j|\mathbf{a}]$ is concave in actions. In comparison, $\mathbb{E}[\sum_{j\in N} a_j|\mathbf{a}]$ is linear in actions. Hence, the optimal action profile is likely to be different depending on the designer's preferences. The analysis proceeds under the assumption that $U_0(\mathbf{a})$ is monotonic.

Definition (AIM Contests): A contest is said to be Assignment Independent and Monotonic (AIM) if $U_0(\mathbf{a})$ is strictly increasing in a_i for all $i \in N$.

The defining feature of any AIM contest is that any optimal action profile must be on the frontier of the set of implementable or feasible action profiles. Thus, the incentive compatibility problem takes centre stage. Therefore, this type of contests is the ideal starting point for understanding the incentive problem and how contest design incentivizes agents.

Two central messages emerge. First, for any frontier action there is an essentially unique assignment rule that is incentive compatible. In other words, incentive compatibility more or less dictates contest design. Thus, for AIM contests, the designer's only real degree of freedom comes from determining which exact frontier action to induce. Second, the fundamental structure of the contest design is the same for all frontier actions. Hence, the principles underlying contest design is the same in all AIM contests.

3.2 Maximal individual and group effort

To understand incentives, it is useful to start by focusing on one given agent in isolation. Given (2), the marginal return to a small increase in a_i is

$$\frac{\partial U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i} = v_i \int \left(\int P_i(q_i, \mathbf{q}_{-i}) L_i(q_i|a_i) g_i(q_i|a_i) dq_i \right) \prod_{j\neq i} g_j(q_j|a_j) d\mathbf{q}_{-i} - 1. \tag{3}$$

Since the expected value of $L_i(q_i|a_i)$ is zero, it follows from the MLRP that $L_i(q_i|a_i)$ is strictly negative for small q_i and strictly positive for large q_i. It is clear that (3) is maximized if the prize is assigned to agent i if and only if $L_i(q_i|a_i)$ is positive. When $L_i(q_i|a_i)$ is positive, a marginal increase in a_i makes it more likely that a performance close to q_i is realized. There is no better carrot than promising the agent a prize for such performances and no better stick than to deny him a prize for performances that become less likely if his action increases.

Proposition 1 *Let $\overline{a}_i$ denote the highest action that agent i can be induced to take. If $\overline{a}_i > 0$ then there is an essentially unique $P_i(q_i, \mathbf{q}_{-i})$ function that induces $\overline{a}_i$.*[9] *This takes the form of a threshold rule,*

$$P_i(q_i, \mathbf{q}_{-i}) = \begin{cases} 1 & \text{if } q_i \geq \widehat{q}_i(\overline{a}_i) \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

where $\widehat{q}_i(a_i)$ denotes the unique value of q_i for which $L_i(q_i|a_i) = 0$.

Proof. See the Appendix. ■

Next, consider a contest in which $n_1 \leq n$ agents are identical to agent 1 ex ante. Let $N_1 \subseteq N$ denote this group of agents. Assume that Ω is symmetric for these agents.[10] Assume moreover that the contest designer is restricted to treating them symmetrically and to inducing the same action for all agents in the group. In fact, it turns out that the optimal way to induce identical actions within the group is to use a symmetric design. Let $\overline{a}_1^s \leq \overline{a}_1$ denote the highest implementable symmetric action in the group and assume that $\overline{a}_1^s > 0$.[11] Define

$$\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s) = \{\omega \in \Omega | \sum\nolimits_{i \in \omega \cap N_1} L_i(q_i|\overline{a}_1^s) \geq \sum\nolimits_{i \in \omega' \cap N_1} L_i(q_i|\overline{a}_1^s) \text{ for all } \omega' \in \Omega\}$$

as the set of assignments where the aggregate likelihood-ratio is maximized within the group, given the performance profile and the action $\overline{a}_1^s$. The assignment rule must then satisfy

$$\sum\nolimits_{\omega \in \Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)} P_\omega(\mathbf{q}) = 1.$$

[9] Here, $P_i(q_i, \mathbf{q}_{-i})$ is "essentially unique" because changes on a set of $\mathbf{q}$ of measure zero are irrelevant. The proof of the proposition outlines a method to characterize $\overline{a}_i$.

[10] Consider $i, j \in N_1$ and assume that there is some $\omega \in \Omega$ with $i \in \omega$ but $j \notin \omega$. Then, there is another $\omega' \in \Omega$ that is identical to ω except that i has been replaced by j.

[11] Roughly speaking this requires that the valuation v_1 is large enough relative to the marginal improvement in the distribution of performance to make active participation worthwhile. This turns out to be the case in the Fullerton and McAfee (1999) model for any $v_1 > 0$.

That is, the assignment must maximize the aggregate likelihood-ratio in the group. The proof is omitted but it follows the same logic as the proof of Proposition 2 in the next section. If rationing is allowed and there are no restrictions on Ω then agents in N_1 with negative likelihood-ratios are excluded, while the agents with the highest positive likelihood-ratios are assigned a prize. This does not uniquely pin down the assignment rule as it does not describe how to assign prizes that are not awarded to agents in the group; $\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)$ typically has more than one element. That is, it does not say whether and which agents outside the group should be awarded a prize, if any are left over.

3.3 Optimal design with two groups

Assume now that there are exactly two groups. There are n_1 contestants that are identical to agent 1 and n_2 that are identical to agent 2, $n_1 + n_2 = n$. Let N_1 and N_2 denote the two groups. Assume that Ω is group-symmetric. The designer is restricted to inducing group-symmetric actions, which again are best achieved by using a group-symmetric assignment rule.[12] The frontier of the feasible set is characterized and interpreted next. Recall that the optimal action profile must be on the frontier in any AIM contest.

A natural starting point is to identify the "corners" of the feasible set. The highest action that agents in group 1 can be induced to take is $\overline{a}_1^s$. To implement this action, the assignment rule must satisfy the criteria described after Proposition 1. In particular, the assignment must belong to $\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)$. Thus, there is less freedom to reward agents in group 2. Let $\underline{a}_2^s$ denote the highest action that agents in group 2 can be induced to take, given agents in group 1 take action $\overline{a}_1^s$. The interesting case is when $\underline{a}_2^s > 0$. Then, the assignment rule must pick one of the assignments in

$$\begin{aligned}\Omega_{N_1,N_2}(\mathbf{q}|\overline{a}_1^s, \underline{a}_2^s) = \{\omega \in \Omega_{N_1}(\mathbf{q}|\overline{a}_1^s) | \textstyle\sum_{i \in \omega \cap N_2} L_i(q_i|\underline{a}_2^s) \\ \geq \textstyle\sum_{i \in \omega' \cap N_2} L_i(q_i|\underline{a}_2^s) \text{ for all } \omega' \in \Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)\}.\end{aligned}$$

[12] This is not restrictive in the case where $n_1 = n_2 = 1$. Extensions to more groups are discussed later but do not add much economic insight. Note that $\pi(\mathbf{q}, \mathbf{a})$ is not restricted to be group-symmetric.

Since it takes all agents across both groups into consideration, $\Omega_{N_1,N_2}(\mathbf{q}|\overline{a}_1^s, \underline{a}_2^s)$ has a unique element almost always. Hence, the assignment rule is essentially unique. This is due to the MLRP which implies that it is a probability zero event that two or more agents have the same likelihood-ratio.[13]

This design has an essentially sequential structure. Whether an agent in group 1 is assigned a prize is completely independent of the performance of agents in group 2. Agents in group 2 fight over what is left after group 1 leaves the table. This design resembles the preference given to Canadian candidates on the academic job market, as described in the introduction. However, the model does not stipulate that the bar should be the same for the two groups, as this is determined where likelihood-ratios are exactly zero in equilibrium.[14]

The parts of the frontier that is not at the corners are more interesting. Here, $a_1 \in (\underline{a}_1^s, \overline{a}_1^s)$ and $a_2 \in (\underline{a}_2^s, \overline{a}_2^s)$, where $\underline{a}_1^s$ and $\overline{a}_2^s$ are defined analogously to $\underline{a}_2^s$ and $\overline{a}_1^s$, respectively. This means that neither groups' equilibrium action in isolation pins down part of the assignment. Thus, there is more design flexibility. The assignment rule must now compromise between giving incentives to both groups of agents simultaneously. Again, it should be clear that the likelihood-ratios play a crucial role, but now they can be compared across agents in the two groups.

Proposition 2 *Consider a contest with two groups, and assume that Ω is group-symmetric and that group-symmetric actions must be implemented. Any action profile $\mathbf{a}$ that is on the frontier of the feasible set with $a_1 \in (\underline{a}_1^s, \overline{a}_1^s)$ and $a_2 \in (\underline{a}_2^s, \overline{a}_2^s)$ is implemented by an assignment rule for which*

$$\sum\nolimits_{\omega \in \Omega(\mathbf{q},\boldsymbol{\mu}|a_1,a_2)} P_\omega(\mathbf{q}) = 1,$$

[13] AIM rules out that the designer cares exclusively about the actions of agents in group 1 (e.g. in-state students). In such a case, implementing $\overline{a}_1^s$ would be optimal but it would not matter which value of $a_2 \leq \underline{a}_2^s$ is implemented alongside $\overline{a}_1^s$. In this case, any optimal action profile is on the boundary – but not necessarily the frontier – of the feasible set. Nevertheless, any optimal assignment rule must have the features discussed at the end of Section 3.2.

[14] However, consider a contest in which $G_i(q_i|a_i) = f_i(a_i)H(q_i) + (1 - f_i(a_i))\,T(q_i)$, where H and T are distribution functions and $f_i(a_i) \in (0,1)$ is increasing and concave. The MLRP and the CDFC are satisfied if H dominates T in terms of the likelihood-ratio. Then, the likelihood-ratio is zero where $H'(q) = T'(q)$, which is independent of the agent's action and identity. Hence, the bar is the same for both groups.

with

$$\Omega(\mathbf{q},\boldsymbol{\mu}|a_1,a_2)=\{\omega\in\Omega|\sum\nolimits_{i\in\omega}\mu_i v_i L_i(q_i|a_i)\geq\sum\nolimits_{i\in\omega'}\mu_i v_i L_i(q_i|a_i) \textit{ for all } \omega'\in\Omega\},$$

and where $\mu_i\in(0,\infty)$*,* $i\in N$*, is endogenously determined and group-symmetric. Since* $\Omega(\mathbf{q},\boldsymbol{\mu}|a_1,a_2)$ *has a unique element almost always, the assignment rule is essentially unique.*

Proof. See the Appendix. ■

As mentioned, in any AIM contest an action profile on the frontier of the feasible set is optimal. Irrespective of the exact form of AIM preferences or the exogenous restrictions on Ω, it is now clear that the structure of the optimal contest is remarkably robust. Specifically, the optimal assignment is determined by a comparison of scaled likelihood-ratios.

The assignment rule in Proposition 2 is fairly intuitive. As discussed in the previous subsection, the power of the incentives facing agent i are determined by the size of $L_i(q_i|a_i)$ when he is assigned a prize. Hence, the μ_i's are calibrated to obtain incentive compatibility across groups. When there are no restrictions on Ω, the assignment rule can be implemented by giving agent i with performance q_i a score of $s_i(q_i)=\mu_i v_i L_i(q_i|a_i)$ and letting the agents with the highest positive scores win a prize but withholding prizes from agents with negative scores.

In other words, $\widehat{q}_i(a_i)$ as defined in Proposition 1 can be seen as a minimum standard for eligibility. At a selective college, this is the minimum admission standard where any student with a lower SAT score is summarily rejected. In a promotion contest, it is the bar that the agent must meet to even be considered for a promotion. In an innovation contest, it is the standard below which the innovation is deemed to be unqualified for consideration, e.g. a drug that fails clinical trials or a reusable spacecraft that is unable to take off twice in two weeks. Note that if the designer can commit to rationing, then rationing must occur with positive probability in equilibrium. Since rationing impacts the design, it is clearly of value to the designer to be able to ration.

To interpret the assignment rule, the case where likelihood-ratios are bounded above is particularly interesting. Then, it must generically hold that $\mu_1 v_1 L_1(\overline{q}_1|a_1)$ and $\mu_2 v_2 L_2(\overline{q}_2|a_2)$ are different in equilibrium. Assume for the sake of argument that the former is larger, and define q_1^t as the threshold where

$$\mu_1 v_1 L_1(q_1^t|a_1) = \mu_2 v_2 L_2(\overline{q}_2|a_2).$$

Thus, an agent in group 1 whose performance exceeds q_1^t is guaranteed to outscore all agents in group 2. Such an agent can never lose to an agent in group 2. Indeed, the agent is guaranteed a prize if $n_1 \leq m$ and Ω is unrestricted. If $n_1 > m$ then the agent may lose but only if he is outscored by sufficiently many agents in his own group. If q_1^t is close to $\overline{q}_1$ then this happens with only a small probability, i.e. the agent is almost guaranteed to win a prize. Thus, this is close to the kind of guaranteed admission that is sometimes given to the best in-state students.

4 Costly and separable contests

The previous section established that the assignment rule is dictated by the agents' incentive compatibility constraints when an action profile on the frontier is implemented. This lack of flexibility may prove costly to the designer if her preferences are not assignment independent. In principle, it may be better to induce an action profile that is not on the frontier of the feasible set. These actions can be induced in many ways, meaning that the assignment rule can now better reflect the designer's objectives.

Thus, the designer's Bernoulli utility is now allowed to depend on the assignment and it is therefore written $\pi_\omega(\mathbf{q}, \mathbf{a})$ in the event that the assignment is ω. This is assumed to be separable in the sense that

$$\pi_\omega(\mathbf{q}, \mathbf{a}) = \pi(\mathbf{q}, \mathbf{a}) + \upsilon_\omega(\mathbf{a}), \quad \omega \in \Omega. \tag{5}$$

Note that preferences are assignment independent if $\upsilon_\omega(\mathbf{a}) = \upsilon(\mathbf{a})$ for all $\omega \in \Omega$. Conversely, separable contests where $\pi(\mathbf{q}, \mathbf{a}) = 0$ is constant is perhaps of special interest. Then, $\pi_\omega(\mathbf{q}, \mathbf{a}) = \upsilon_\omega(\mathbf{a})$ captures *any* contest setting where $\mathbf{q}$ is a signal that is worthless on its own, but which is informative about actions. Much of the contest literature is of course precisely concerned with objective functions that depend exclusively on actions.

The assignment and the action profile may interact. As an example, $v_\omega(\mathbf{a}) = \sum_{i\in\omega} a_i$ captures a case where the designer only cares about the actions of the winners. For a firm, these might be the employees that are promoted or retained in the organization after a trial period or internship. Similarly, a university is likely to be more interested in the effort undertaken by any given student to prepare for university in the event that he is admitted than if he is not.

It is also assumed that preferences are monotonic, or more precisely that for any performance profile and any assignment it holds that

$$\frac{\partial\left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right)}{\partial a_i} \geq 0 \text{ for all } i \in N \text{ with } \frac{\partial\left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right)}{\partial a_i} > 0 \text{ for all } i \in \omega, \tag{6}$$

where $U_0(\mathbf{a}) = \mathbb{E}[\pi(\mathbf{q}, \mathbf{a})|\mathbf{a}]$ as before. Thus, for any given assignment, the designer is better off if agents take higher actions.

Definition (SEP-M Contests): A contest is Separable and Monotonic (SEP-M) if (5) and (6) hold.

The next example provides another illustration of the flexibility of SEP-M contests. It also demonstrates that the second-best or optimal action profile is not necessarily on the frontier of the feasible set even when $\pi_\omega(\mathbf{q}, \mathbf{a})$ is strictly increasing in all actions.

Example 1.I (Social welfare versus effort maximization): Consider a contest with a single prize that must be allocated. The designer cares at least in part about social welfare. For instance, a politician may care both about social welfare and the benefits he derives personally from the lobbyists' efforts to persuade him to adopt one particular policy. Since the utility of a winning agent is $v_i - a_i$ and the utility of a losing agent is $-a_i$, this can be captured by

$$\begin{aligned} \pi_{\{i\}}(\mathbf{q}, \mathbf{a}) &= \gamma\left(v_i - \textstyle\sum_{j\in N} a_j\right) + (1-\gamma)\textstyle\sum_{j\in N} a_j \\ &= \gamma v_i + (1-2\gamma)\textstyle\sum_{j\in N} a_j \end{aligned} \tag{7}$$

for $i \in N$, where $\gamma \in (0, 1)$ is the weight assigned to agents' utility. Note that this is a separable contest. Epstein, Mealem, and Nitzan (2011) describe a contest much like this; their approach is reevaluated in Section 5.

Assume that $v_1 > v_2 \geq v_3 \geq ... \geq v_n$. Based on (7), the optimal design is trivially to assign the prize to agent 1 if $\gamma > \frac{1}{2}$, regardless of the performance profile. In this case, no agent wastes effort in equilibrium. The case where $\gamma < \frac{1}{2}$ is more interesting, as the conflict between social welfare and the private rewards to the designer then presents a real trade-off. The contest is now a SEP-M contest. Assume that $\overline{a}_i > 0$ for all $i \in N$. Imagine inducing an action profile on the frontier of the feasible set. By definition, aggregate effort is then below $\sum_{j\in N} \overline{a}_j$. At the same time, agent 1 must win with a probability bounded away from 1. Otherwise, it is impossible to induce effort from any agent, contradicting the premise that the action profile is on the frontier. Let $t < 1$ denote any such upper bound. Then, expected utility is bounded above by

$$\gamma \left(tv_1 + (1-t)\, v_2\right) + (1-2\gamma) \textstyle\sum_{j\in N} \overline{a}_j. \tag{8}$$

Now let v_1 increase while simultaneously changing G_1 in such a way that $\overline{a}_1$ and t are unchanged (the best-shot model in Section 5 describes a parameterized distribution that can be manipulated to achieve this). When v_1 becomes large enough, (8) is strictly smaller than γv_1, which is the utility that the designer can derive from just giving the prize to the agent for sure. Hence, no frontier action can be optimal even though the designer's Bernoulli utility is strictly increasing in all actions. $\square$

As before, the optimal design is characterized for contests with two groups of agents. For concreteness and brevity, focus is also on contests where any second-best action profile is interior, or $a_i > 0$ for all $i \in N$, but not on the frontier of the feasible set. That is not to say that the second-best action cannot be on the frontier of the feasible set, but in that case the design must take the form in Proposition 2.

Proposition 3 *Consider a SEP-M contest with two groups, and assume that Ω is group-symmetric and that group-symmetric actions must be implemented. Any second-best action profile* $\mathbf{a}$ *that is interior and not on the frontier of the feasible set is optimally implemented by an assignment rule for which*

$$\textstyle\sum_{\omega\in\Omega_{SEP-M}(\mathbf{q},\boldsymbol{\mu}|a_1,a_2)} P_\omega(\mathbf{q}) = 1$$

with

$$\Omega_{SEP-M}(\mathbf{q},\boldsymbol{\mu}|a_1,a_2)=\{\omega\in\Omega|\upsilon_\omega(\mathbf{a})+\sum\nolimits_{i\in\omega}\mu_i v_i L_i(q_i|a_i)$$
$$\geq \upsilon_{\omega'}(\mathbf{a})+\sum\nolimits_{i\in\omega'}\mu_i v_i L_i(q_i|a_i)\ \textit{for all}\ \omega'\in\Omega\}$$

and where $\mu_i\in(0,\infty)$*,* $i\in N$*, is endogenously determined and group-symmetric.*

Proof. See the Appendix. ■

The presence of the $\upsilon_\omega(\mathbf{a})$ term in the designer's utility implies that she has an ex post interest in the assignment. Ideally, the designer would prefer to select an assignment that maximizes $\upsilon_\omega(\mathbf{a})$. However, this may not be incentive compatible. Thus, each possible assignment, ω, is assigned an aggregate score, $\upsilon_\omega(\mathbf{a})+\sum_{i\in\omega}\mu_i v_i L_i(q_i|a_i)$. This score reflects a compromise between two considerations. First, $\upsilon_\omega(\mathbf{a})$ is directly relevant to the designer but as mentioned this does not take incentive compatibility into account. Hence, the second term, $\sum_{i\in\omega}\mu_i v_i L_i(q_i|a_i)$, is present to "fix" the problem and ensure that active agents' first-order conditions are satisfied.[15]

Compared to Proposition 2, each assignment is thus given an additive bonus, $\upsilon_\omega(\mathbf{a})$, that is independent of the performance profile. This bonus reflects whatever preference the designer has over the identity and composition of the winners. Recall that this is different from any legal restrictions placed on Ω which limits the final assignment. Consider Example 1 again, and assume that there are two agents and $\gamma\in(0,\frac{1}{2})$. It is intuitive that both agents should be induced to be active as long as γ is small or the agents are not too asymmetric. In this case, the designer cares a bit but not too much about which agent is awarded the prize. Then, agent i is simply given a score of $\gamma v_i+\mu_i v_i L_i(q_i|a_i)$.

The next example shows how the SEP-M model applies to contest in which the prizes are costly to the designer. Other things equal, she would rather not award any prizes ex post, but this would create an incentive problem ex ante.

[15] Action profiles near the frontier of the feasible set presumably have very large μ_i's. As the μ_i's goes to infinity, the $\pi_\omega(\mathbf{q},\mathbf{a})$ terms lose their significance and the assignment rule converge to that in Proposition 2. Note that the μ_i's can go to infinity while their ratios converge to the ratios implied by Proposition 2.

Example 2.I (Costly prizes and rationing): Consider a contest in which

$$\pi_\omega(\mathbf{q}, \mathbf{a}) = \textstyle\sum_{i \in N} a_i - C(|\omega|),$$

where $C(\cdot)$ is an increasing and convex cost function that measures how costly it is to the designer to award $|\omega|$ prizes, with $C(0) = 0$. Rationing is assumed to be possible and there are no restrictions on Ω except for the restriction that at most m prizes can be allocated. However, for this example allow $m = n$. Since the designer is then free to decide how many prizes to assign, the optimal number of prizes is effectively endogenized.

With two groups of agents that are both active, the assignment ω obtains a score of

$$\sum_{i \in \omega} \mu_i v_i L_i(q_i|a_i) - C(|\omega|).$$

Hence, among all assignments in which a fixed number of prizes are awarded, the assignment with the highest score is the one in which the agents with the highest $\mu_i v_i L_i(q_i|a_i)$ are awarded a prize. Thus, agents can be arranged in descending order of their values of $\mu_i v_i L_i(q_i|a_i)$. If x prizes are awarded, they are awarded to the first x agents in line. Now compare two assignments ω and ω' that consist of the agents with the x and $x' = x + 1$ highest values of $\mu_i v_i L_i(q_i|a_i)$, respectively. Let agent j be the agent who is in ω' but not in ω. The additional prize should be awarded if and only ω' scores higher than ω, or

$$\mu_j v_j L_j(q_j|a_j) > C(x+1) - C(x).$$

Thus, in equilibrium it is straightforward to determine the number of prizes to be awarded and the identity of the winners.

There are lucky performance profiles where many or perhaps every agent is awarded a prize. On the other hand, there are also performance profiles where all agents have negative likelihood-ratios, in which case no prizes are awarded. In short, the designer should not commit to awarding a fixed number of prizes and the number of prizes that are actually awarded is stochastic ex ante.

The same analysis applies if $\pi_\omega(\mathbf{q}, \mathbf{a}) = \sum_{i \in \omega} a_i - C(|\omega|)$, except that in this case the marginal score of agent j is $a_j + \mu_j v_j L_j(q_j|a_j)$. $\square$

5 Revisiting lottery contests

This section utilizes two different model specifications to illustrate some of the main results. These model specifications also make it possible to reexamine some of the key results from the contest literature that relies on variants of the lottery CSF. Thus, it is assumed throughout that there is a single prize.

The best-shot model: Assume that agent i's distribution function can be written

$$G_i(q|a_i) = H_i(q)^{f_i(a_i)},\ q \in [\underline{q}_i, \overline{q}_i], \tag{9}$$

for all $i \in N$, where $H_i(q)$ is a distribution function with density $h_i(q)$. Thus, $G_i(q_i|a_i)$ is the distribution of the best draw from $H_i(q)$ – the best-shot – out of a total of $f_i(a_i) \geq 0$ draws. In an innovation contest, H_i can be interpreted as the distribution of the quality of a single idea and f_i as the number of ideas. The MLRP and the CDFC are both satisfied as long as $f_i'(a_i) > 0 \geq f_i''(a_i)$.

The setting is inspired by Fullerton and McAfee (1999). However, they assume that $H_i(q) = H(q)$ for all $i \in N$. In words, all agents have ideas that are equally good ex ante but some agents may have more ideas than others. In this case, agent i wins an unbiased contest with an ex ante probability of

$$p_i(\mathbf{a}) = \frac{f_i(a_i)}{\sum_{j=1}^n f_j(a_j)}, \tag{10}$$

when $\sum_{j=1}^n f_j(a_j) > 0$. This is intuitive. After all, agent i has $f_i(a_i)$ ideas out of a total of $\sum_{i\in N} f_j(a_j)$ idea. Each idea has an equal chance of being the best idea, thus yielding the CSF in (10). Hence, this produces the popular lottery CSF.

However, (10) does not obtain when the H_i's are allowed to be heterogenous. The reason is that different agents now draw ideas that are of different quality ex ante. Such a setting thus cannot be analyzed using (10) but it succumbs to the approach suggested in this paper. Henceforth, the "best-shot model" refers to (9) with potentially heterogenous H_i's. The special case in which all H_i's are identical is referred to as the Fullerton and McAfee (1999) model. □

The exponential-noise model: The exponential-noise model is due to Hirschleifer and Riley (1992). Importantly, it is assumed that there are precisely two agents.

Given a_i, agent i's observable performance is exponentially distributed with mean $f_i(a_i) > 0$ when $a_i > 0$, $i = 1, 2$. That is, the distribution function is

$$G_i(q|a_i) = 1 - e^{-\frac{q}{f_i(a_i)}},\ q \in [0, \infty). \tag{11}$$

Equivalently, performance is impacted by multiplicative noise such that $q_i = f_i(a_i)\varepsilon_i$, where ε_i is exponentially distributed with mean one. Hirschleifer and Riley (1992) showed that this yields the lottery CSF in an unbiased contest.

Since $f_i(a_i)$ is the expected value of agent i's performance, it is natural to assume that $f_i'(a_i) > 0$ and $f_i''(a_i) \leq 0$. The MLRP is then satisfied but the CDFC is not. Nevertheless, it will be established that the agents' first-order conditions are sufficient for incentive compatibility along the frontier of the feasible set under the assumption that

$$f_i(a_i) = \beta_i a_i^{r_i}, \tag{12}$$

where $\beta_i > 0$, $r_i \in (0, 1]$, $i = 1, 2$. This is the most commonly used impact function used in the contest literature. $\square$

5.1 Two sides of the same coin

This subsection contrasts the best-shot and exponential noise models under the assumption that (12) holds and that $n = 2$. In keeping with most of the literature, rationing is ruled out to start and the contest is assumed to be an AIM contest. Thus, the frontier of the feasible set is of interest. Some details are relegated to Appendix B, where both models are considered in more generality.

The two models turn out to be in some ways on opposite sides of the same coin. To begin, compare the adjusted scores in the two models. If $\mathbf{a}^*$ is the equilibrium action profile, agent i's score in the best-shot model is

$$\begin{aligned} s_i(q_i) &= \mu_i v_i L_i(q_i|a_i^*) \\ &= \tau_i \left(1 + \ln G_i(q_i|a_i^*)\right), \end{aligned} \tag{13}$$

where

$$\tau_i = \mu_i v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)}.$$

In the exponential noise model, agent i's score is

$$\begin{aligned} s_i(q_i) &= \mu_i v_i L_i(q_i|a_i^*) \\ &= -\tau_i \left(1 + \ln\left(1 - G_i(q_i|a_i^*)\right)\right), \end{aligned} \tag{14}$$

where τ_i is defined as before. In either case, the optimal assignment rule can be implemented as follows. First, q_i is translated into the quantile where it sits in the equilibrium distribution $G_i(q_i|a_i^*)$. This intermediate score then undergoes a monotonic transformation before being multiplied by an endogenous and identity-dependent constant, τ_i. The winner is the agent with the highest final score.

In the two-agent case only the relative sizes of τ_1 and τ_2 matter. Thus, define $c = \frac{\tau_2}{\tau_1}$. For any given $c \in [0, \infty)$, it is possible to derive the action profile that is being implemented. The entire frontier can thus be traced out by letting c run from 0 to ∞.[16] Remarkably, the frontiers coincide in the two models.

Proposition 4 *Consider the two-agent best-shot or exponential noise model and assume that $f_i(a_i) = \beta_i a_i^{r_i}$, $\beta_i > 0$, $r_i \in (0, 1]$, $i = 1, 2$. When rationing is ruled out, the frontier of the feasible set contains the corners $(\overline{a}_1, 0)$ and $(0, \overline{a}_2)$, where*

$$\overline{a}_i = \frac{r_i v_i}{e}. \tag{15}$$

Any interior action profile $\mathbf{a}^* = (a_1^*, a_2^*)$ is on the frontier if and only if

$$a_1^* = r_1 v_1 F\left(\frac{1}{c}\right) \text{ and } a_2^* = r_2 v_2 F(c) \tag{16}$$

for some $c \in (0, \infty)$, where $c = \frac{\tau_2}{\tau_1}$ and

$$F(c) = \begin{cases} \frac{c}{(1+c)^2} e^{c-1} & \text{if } c \in (0,1) \\ \frac{c^2+c-1}{(1+c)^2} e^{\frac{1-c}{c}} & \text{if } c \geq 1 \end{cases}.$$

Here, $F(c)$ is strictly increasing in c and satisfies $F(1) = \frac{1}{4}$. Hence, a_2^* is increasing in c and a_1^* is decreasing in c.

[16] The characterization result can be extended to any increasing and concave $f_i(a_i)$ in the best-shot model (see Appendix B).

Proof. See the Appendix. ■

Note that (15) and (16) do not depend on H_1 and H_2 in the best-shot model, nor do they require $H_1 = H_2$. The designer can always transform q_i into the quantile $\widetilde{q}_i = H_i(q_i)$ and use this as the basis for contest design.[17] Given a_i, the distribution of $\widetilde{q}_i$ is $\widetilde{q}_i^{f_i(a_i)}$, $\widetilde{q}_i \in [0,1]$, independently of what $H_i(q_i)$ is. It follows that the set of implementable action profiles is independent of H_1 and H_2.

It is now possible to derive the optimal action profile and thus the optimal assignment rule in any AIM contest. The next result illustrates this under the assumption that $\pi(\mathbf{q},\mathbf{a}) = a_1 + a_2$, which is the leading example in the literature.

Proposition 5 *Consider the two-agent best-shot or exponential noise model and assume that $f_i(a_i) = \beta_i a_i^{r_i}$, $\beta_i > 0$, $r_i \in (0,1]$, $i = 1,2$. Let $k = \frac{r_2 v_2}{r_1 v_1}$ and assume that $k \geq 1$ Finally, assume that the prize has to be awarded. Then, total effort, $a_1 + a_2$, is maximized by letting $\frac{\tau_2}{\tau_1} = \frac{r_2 v_2}{r_1 v_1}$ (or $c = k$). In equilibrium, actions are*

$$\begin{aligned} a_1 &= r_1 \frac{k}{(k+1)^2} e^{-\frac{1}{k}(k-1)} v_1 = r_2 \frac{1}{(k+1)^2} e^{-\frac{1}{k}(k-1)} v_2 \\ a_2 &= r_2 \frac{k^2+k-1}{(k+1)^2} e^{-\frac{1}{k}(k-1)} v_2 \end{aligned}$$

and total effort is

$$a_1 + a_2 = \frac{k}{k+1} e^{-\frac{1}{k}(k-1)} r_2 v_2. \tag{17}$$

As an alternative objective, consider $\pi(\mathbf{q},\mathbf{a}) = q_1 + q_2$. Then, the designer's problem is to maximize the total expected performance, $\mathbb{E}[q_1|a_1] + \mathbb{E}[q_2|a_2]$. This is $f_1(a_1) + f_2(a_2)$ in the exponential model, but in the best-shot model it also depends on H_1 and H_2. Hence, the optimal design is sensitive to the model. This may sound unsurprising but that is because the roles of the distribution function and the performance profile are made explicit here. After all, whenever $\pi(\mathbf{q},\mathbf{a})$ is assumed to depend only on $\mathbf{a}$ – as is implicitly the case in most papers that rely on the lottery CSF – then the optimal action profile must be the same in the two models, as illustrated in the previous proposition.

[17] It is also for this reason that β_i does not appear in (16). After all, the distribution can be written as $G_i(q_i|a_i) = \left(H_i(q_i)^{\beta_i}\right)^{a_i^{r_i}}$. Hence, changing β_i is similar to changing H_i.

There are a number of other differences between the models as well. Compare (13) to (14) once more. First, the range of the agent i's score is $[-\tau_i, \infty)$ in the former but $(-\infty, \tau_i]$ in the latter. Hence, a larger τ_i moves the range of scores in opposite directions. Another difference is that (13) relies on $G_i(q_i|a_i^*)$ whereas (14) relies on $1 - G_i(q_i|a_i^*)$. For these reasons, winning probabilities move in opposite direction along the frontier of the feasible set.

Proposition 6 *Consider the two-agent best-shot or exponential-noise model and assume that $f_i(a_i) = \beta_i a_i^{r_i}$, $\beta_i > 0$, $r_i \in (0,1]$, $i = 1,2$. Along the frontier of the feasible set when rationing is ruled out, agent 2's equilibrium winning probability is $W(c)$ in the best-shot model but $1 - W(c)$ in the exponential-noise model, where*

$$W(c) = \begin{cases} \frac{1}{1+c} e^{c-1} & \text{if } c \in (0,1) \\ 1 - \frac{c}{1+c} e^{\frac{1-c}{c}} & \text{if } c \geq 1 \end{cases}$$

and $c = \frac{\tau_2}{\tau_1}$. Here, $W(c)$ is strictly increasing in c and satisfies $W(0) = e^{-1} = 0.368$, $W(1) = \frac{1}{2}$, and $\lim_{c \to \infty} W(c) = 1 - e^{-1} = 0.632$.

Proof. See the Appendix. ■

Proposition 6 implies that higher $c = \frac{\tau_2}{\tau_1}$ favors agent 2 in the best-shot model but does the opposite in the exponential noise model, at least in terms of equilibrium winning probabilities. Combining Propositions 5 and 6, note that it is *not* optimal to level the playing field when the objective is to maximize $a_1 + a_2$. In fact, Proposition 6 implies that the identity of the agent who is favored *differ* in the best-shot and exponential-noise model. In short, the stochastic performance approach does not have an unambiguous policy implication in this regard. Again, the details of the performance distributions matter.

The analysis has ignored rationing. In AIM contests with rationing, agent i has zero chance of winning the prize if $q_i < \widehat{q}_i(a_i)$. In the best-shot model, (13) implies that $\widehat{q}_i(a_i)$ satisfies $G_i(\widehat{q}_i(a_i)|a_i) = e^{-1}$. Thus, there is a fixed chance, $e^{-1} = 0.368$, that any given agent will be disqualified for failing to meet the standard. In a sense, the standard is the same for all agents in "probabilistic" terms. Appendix B extends Propositions 4 and 5 to allow rationing in the best-shot model. It remains the case that $\frac{\tau_2}{\tau_1} = \frac{r_2 v_2}{r_1 v_1}$ (or $c = k$) maximizes total effort. However, due to the extra threat of rationing, both agents work harder.

5.2 Contest success functions

Given instruments τ_1 and τ_2, it is possible to derive the implied contest success functions. As is demonstrated in Appendix B, these CSFs turn out *not* to be lottery CSFs. Thus, even if the starting point is an unbiased lottery contest as in the Fullerton and McAfee (1999) or exponential-noise models, endogenizing the contest design fundamentally alters the resulting CSF.[18]

This last point clashes with a popular approach in the existing contest literature. It is common to assume that the designer can implement a CSF that is some variant of

$$p_i(\mathbf{a}|\boldsymbol{\delta}, \mathbf{b}, z) = \frac{b_i f_i(a_i) + \delta_i}{\sum_{j=1}^{n} (b_j f_j(a_j) + \delta_j) + z}. \tag{18}$$

Here, one or more of $\boldsymbol{\delta} = (\delta_1, ..., \delta_n)$, $\mathbf{b} = (b_1, ..., b_n)$, and z are considered to be design instruments. Typically, each variable is restricted to be non-negative. The interpretation of δ_i is that it captures agent i's head start, while b_i is a multiplicative bias or handicap. Thus, it is as if agent i earns $b_i f_i(a_i) + \delta_i$ lottery tickets. Finally, z can be thought of as the number of lottery tickets that the designer reserves for herself, thus admitting the possibility of rationing.

Fullerton and McAfee's (1999) model provides a microfoundation for the unbiased lottery CSF, $p_i(\mathbf{a}|\mathbf{0}, \mathbf{1}, 0)$. However, it is hard to reconcile (18) with this model, where actions are presumably unobservable.[19] If actions are unobservable, then how are biases and head starts applied to $f_i(a_i)$ in order to calculate the new number of tickets? The problem is that (18) appear to treat the CSF as the primitive. However, it is the performance technology G_i that is the primitive. The CSF is just a reduced form that integrates out the uncertainty over q.

Another way of expressing the problem is that the literature has not provided a microfoundation for (18). Nevertheless, it turns out to be possible to do so.

Proposition 7 *Consider the Fullerton and McAfee (1999) model. Assign agent i with performance q_i a base score of $s_i^B(q_i) = H(q_i)^{1/b_i} \in [0, 1]$, $b_i > 0$. Draw*

[18] There are also axiomatic justifications for the lottery CSF, see Skaperdas (1996) and Clark and Riis (1998). However, once contest design is endogenized, there appear little reason to think that the designer will voluntarily limit herself to contests that satisfy nice axioms.

[19] If actions are observable, then an auction-like mechanism is likely to be preferable to a lottery contest.

an auxiliary score s_i^{AUX} *for agent i from the distribution* $\left(s_i^{AUX}\right)^{\delta_i}$, $s_i^{AUX} \in [0,1]$, $\delta_i \geq 0$. *Let agent i's final score be* $s_i^{FM}(q_i) = \max\{s_i^B(q_i), s_i^{AUX}\}$. *Finally, draw a score* s^D *for the designer from the distribution* $\left(s^D\right)^z$, $s^D \in [0,1]$, $z \geq 0$. *Let the individual (agent or designer) with the highest score win. Then, the CSF is given by (18).*

Proof. See the Appendix. ■

The transformation of q_i into a base score maps the idea from the support $[\underline{q}, \overline{q}]$ into a quality index on $[0,1]$, where the index is identity dependent via b_i. Given action a_i, agent i then draws $b_i f_i(a_i)$ ideas from a uniform distribution on this index. He is then given δ_i fake ideas by the designer, again drawn from a uniform distribution. The agent now has a total of $b_i f_i(a_i) + \delta_i$ real and fake ideas. The designer also draws z fake ideas from a uniform distribution. Each idea, real or fake, has an equal chance of winning, yielding (18).

The stochastic nature of the fake ideas may or may not be palatable. Thus, Proposition 7 should not be taken as a defense of (18) but rather as a clarification of the lengths one must go to in order to justify it. The transformation of the performance into a quality index seems more appealing. However, this particular transformation is still ad hoc.[20] In fact, Proposition 7 merely shows that (18) can be implemented in the Fullerton and McAfee (1999) model. Hence, it follows that the set of implementable actions must in reality be strictly larger than the set of actions that can be implemented by using (18).[21]

One drawback of using (18) for contest design is that it says little about how to implement the optimal design in practice. For instance, how exactly is the playing field supposed to be made level if the designer does not observe actions? Proposition 7 tells us how this can be achieved by linking design to the observable signals. In other words, the kind of story embodied in Proposition 7 is important if the desire is to apply lessons from (18) in practice. The issue is that (18) pushes the performance profile to the back, which is unfortunate since this is the

[20] Similarly, giving agents a multiplicative bonus in the exponential noise model yields $p_i(\mathbf{a}|\mathbf{0}, \mathbf{b}, 0)$. This can also be obtained by variying the β_i parameter in Clark and Riis' (1996) random utility framework. Again, these are ad hoc ways to manipulate the contest.

[21] Fu and Wu (2020) and most of the prior literature restrict head starts to be non-negative. Drugov and Ryvkin (2017) show that negative head starts may be better. However, negative head starts cannot be justified by Proposition 7.

observable variable. The stochastic performance approach in the current paper has the distinct advantage that it starts directly from the observables.

The following three examples contrast the two approaches. The gain from designing a fully optimal contest can be sizeable and welfare and policy implications may be substantially different as well.

Example 1.II (Social welfare versus private rewards): Epstein, Mealem, and Nitzan (2011) describe a contest much like the one described in Example 1.I. They assume that there are only two contestants, with $v_1 > v_2 > 0$. They consider a version of (18) with $z = 0$ and $\boldsymbol{\delta} = \mathbf{0}$ but allow $\mathbf{b}$ to be a design instrument. Assuming that $f(a_i) = a_i^r$, $r \in (0, 1]$, Epstein, Mealem, and Nitzan (2011) show that if γ exceeds a threshold $\gamma_0 \in \left(0, \frac{1}{2}\right)$ then $b_2 = 0$ is optimal. In this case, agent 1 wins with probability one and neither agent exerts any effort. If $\gamma < \gamma_0$ then $b_1, b_2 > 0$ and both agents exert effort in equilibrium.

A similar conclusion obtains in the current paper but with some quantitative differences. Proposition 7 implies that (18) underestimates the expected payoff of inducing both agents to be active in the best-shot or exponential-noise models. The reason is that the designer has more flexibility than what is suggested by (18). Hence, it is optimal to induce active participation by both agents for more values of γ. In short, the current model predicts that the cut-off γ_0 is higher. In other words, when complete design flexibility is allowed, it is optimal less often to completely favor one agent to the exclusion of the other agent. $\square$

Example 2.II (Rationing with costly prizes): Dasgupta and Nti (1998) consider a symmetric model with $n \geq 2$ agents who all assign the same value, v, to the prize. The designer's own-use valuation is $v_0 \geq 0$ but she always benefits from the sum of actions. In the current paper's terminology, $\pi_{\{i\}}(\mathbf{q}, \mathbf{a}) = \sum_{j \in N} a_j$, $i \in N$, and $\pi_{\emptyset}(\mathbf{q}, \mathbf{a}) = v_0 + \sum_{j \in N} a_j$. This setup is isomorphic to Example 2.I, when $m = 1$ and $C(1) = v_0$.

Dasgupta and Nti (1998) model the possibility of rationing by adopting (18) and allowing $z > 0$. Here, $\boldsymbol{\delta} = \mathbf{0}$ is optimal. Hence, they derive the optimal $(\mathbf{b}, z)$ combination and conclude that $z = 0$ when v_0 is sufficiently small. That is, the prize should never be withheld if the designer's own-use value is low enough.

However, Proposition 3 and Example 2.I imply that rationing is always part of the fully optimal design. Indeed, this conclusion holds for any $(G_1, ..., G_n)$ as long as the MLRP and the CDFC are satisfied. $\square$

Example 3 (Head starts and Handicaps in AIM contests): An active literature relies on (18) to derive optimal head starts and handicaps, usually with the assumption that $z = 0$. To facilitate comparison with the previous results, consider an $n = 2$ agent contest with $f_i(a_i) = \beta_i a_i^{r_i}$, $\beta_i > 0$, $r_i \in (0, 1]$. In the confines of (18), Fu and Wu (2020) show that individual actions are maximized by perfectly levelling the playing field such that each agent wins with probability 0.5.[22] Since individual actions are maximized, this design is optimal whenever the designer's expected utility is strictly increasing in actions. In the ensuing equilibrium, $a_i = \frac{r_i v_i}{4}$, $i = 1, 2$, and $a_1 + a_2 = \frac{1+k}{4k} r_2 v_2$, where $k = \frac{r_2 v_2}{r_1 v_1}$ is defined as before. Incidentally, note that the same outcome is obtained in (16) by letting $c = \frac{\tau_2}{\tau_1} = 1$. However, using the optimal design from Proposition 5 yields $a_1 + a_2 = \frac{k}{k+1} e^{-\frac{1}{k}(k-1)} r_2 v_2$ by (17). The percentage improvement is increasing in the level of asymmetry, k, and converges to $\frac{4-e}{e}$ – or just above 47% – as $k \to \infty$. Hence, using the correct design can lead to a substantial increase in the designer's payoff compared to what is suggested by the existing literature that does not utilize the microfoundations. Moreover, recall that the discussion after Proposition 6 also reveals that a perfectly level playing field is not optimal. $\square$

6 Discussion

6.1 More groups

The general logic behind optimal contest design remains the same when there are more than two groups of agents or when identical agents do not have to be treated symmetrically. However, it is more cumbersome to describe the frontier of the feasible set. For instance, with 3 groups the first group could be induced to take action $\overline{a}_1^s$ and the second to take action $\underline{a}_2^s$. The third group can then be incentivized only by picking among the set of assignments in $\Omega_{N_1,N_2}(\mathbf{q}|\overline{a}_1^s, \underline{a}_2^s)$.[23]

[22] For other recent papers in this literature, see Franke (2012), Franke, Leininger, and Schwartz (2013), and Franke, Leininger, and Wasser (2018).

[23] With more groups, $\Omega_{N_1,N_2}(\mathbf{q}|\overline{a}_1^s, \underline{a}_2^s)$ does not take all agents into consideration and it therefore generally speaking has multiple elements.

Alternatively, group 1 can be induced to take action $\overline{a}_1^s$, while groups 2 and 3 are jointly incentivized by selecting assignments in $\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)$ that balance incentives between those groups. From the point of view of groups 2 and 3, the design would look like the design in Proposition 2, except Ω is replaced by $\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)$.

It is also possible that groups 1 and 2 are incentivized first using a design as in Proposition 2, and that group 3 is then incentivized by selecting more carefully among the assignments in $\Omega(\mathbf{q}, \boldsymbol{\mu}|a_1, a_2)$. Finally, all three groups can jointly be induced to take intermediate actions by balancing incentives between all three groups simultaneously. In the latter case, this is achieved by using a design exactly as in Proposition 2, but with more groups.

In general, then, the agents or groups are assembled into "blocks" that are then placed in a "queue". Incentivizing the first block cuts down the set of assignments from Ω to something like $\Omega_{N_1}(\mathbf{q}|\overline{a}_1^s)$ or $\Omega(\mathbf{q}, \boldsymbol{\mu}|a_1, a_2)$. Incentivizing the second block further narrows down the set of assignments that can subsequently be used to incentivize the third block, and so on, in a process similar to that described in the first paragraph. If all agents or groups are jointly incentivized, then the design takes the form in Proposition 2, just with more groups.

It is possible that the process grinds to a halt in the sense that early blocks narrow down Ω so much that there is no degrees of freedom left to incentivize later blocks. For instance, assume that there is a single prize and two agents, $m = 1$ and $n = 2$, and that rationing is ruled out, or $\Omega = \{\{1\}, \{2\}\}$. Imagine that agent 1 is presented with a threshold rule in order to incentivize action $\overline{a}_1$. If agent 1 does not meet the threshold, then agent 2 has to be awarded the prize since rationing is ruled out. Hence, it is impossible to incentive agent 2, and his best response is $a_2 = 0$. Here, even though rationing is ruled out, agent 1 is incentivized by the threat that the prize may be given to agent 2.

6.2 Scoring rules, prize splitting, and entry fees

Lazear and Rosen (1981) consider an extension to tournaments in which one agent is given a head start. See also Fain (2009). In other words, each agent has a linear scoring rule. However, since likelihood-ratios are rarely linear in q_i, this is quite unlikely to be optimal. Indeed, even in cases where likelihood-ratios are linear, the scoring functions are unlikely to have the same slope. In the classic moral hazard literature, Jewitt (1988) argue that the likelihood-ratio is often concave in performance. For instance, this is the case in the best-shot model when H_i is log-concave, which is true of most commonly used distribution functions.

Similarly, Nalebuff and Stiglitz (1983) and Imhof and Kräkel (2006) allow the prize to be split among agents if performances are close. Since this is not a deterministic assignment rule, such a policy is not optimal in the current setting.

However, this ignores entry fees. Entry fees can be added to the model. Optimal entry fees are identity-dependent and force the participation constraints to bind. For AIM contests with a single and common monetary prize of value v, the size of v can likewise be endogenized. Increasing the prize expands the feasible set. Given risk neutrality and endogenous entry fees and prizes, the first-best can be achieved in AIM contests. Simply expand the prize or feasible set enough to implement the first-best solution to $U_0(\mathbf{a}) - \sum_{i \in N} a_i$ and then extract all rent via entry fees. However, this may not be feasible outside AIM contests.[24] Likewise, there are contests where the prize is not monetary, or homogenous, or easily adjustable.

6.3 Richer models of stochastic performance

It has been assumed that signals are independent. This assumption is built into existing microfoundations for the generalized lottery CSF but there is no conceptual reason to insist on this assumption more generally. However, the incentive compatibility problem becomes more complicated when correlation is permitted. This technical issue is left for future research.

Another simplifying assumption is that q_i is one-dimensional. In some applications, it may be more reasonable to assume that q_i is a vector. The analysis leading to Propositions 1–3 still applies, meaning that the optimal assignment rule keeps the same features. To understand this, note that the likelihood-ratio is a scalar even if performance is multi-dimensional. Hence, comparing likelihood-

[24] Consider $\pi_\omega(\mathbf{q}, \mathbf{a}) = t$ if $\omega = \emptyset$, $\pi_\omega(\mathbf{q}, \mathbf{a}) = f(a_1) + t - v$ if $\omega = \{1\}$ and $\pi_\omega(\mathbf{q}, \mathbf{a}) = t - v$ otherwise, where $f(\cdot)$ is a concave production function, t is the total entry fee collected from agents, and v is the size of the prize. The first-best solves $f'(a_1) = 1$ but it must also involve $P_{\{1\}}(\mathbf{q}) = 1$ for all $\mathbf{q}$. The latter cannot incentivize the former.

ratios remain key. Checking incentive compatibility of the assignment rule may be more complicated, however. See Conlon (2009) and Kirkegaard (2017) for ways of justifying the first-order approach with many-dimensional signals. If the likelihood-ratios are increasing along each performance dimension, then the condition in Proposition 2 is satisfied on an increasing set, in the sense of Conlon (2009). If distribution functions satisfy his CISP condition, then the first-order conditions are sufficient. See Jung and Kim (2015) for an alternative approach that is founded more directly on the distribution of the likelihood-ratios.

7 Conclusion

This paper pursues a model of contests that is based on stochastic performance. Contest design is then a team moral hazard problem in which the assignment rule is manipulated to incentivize effort. The principles behind optimal design are remarkably robust to both the designer's objectives and the distributions of performance. Consistent with the standard single-agent principal-agent model, likelihood-ratios play a key role in determining an agent's compensation or, in this case, the probability that he wins a prize.

The model provides both practical and conceptual insights. The optimal design is consistent with guaranteed admissions policies and heterogeneous admission standards. It endogenizes standards for eligibility and explains why the number of prizes that are awarded may be stochastic ex ante. Conceptually, the approach offers an alternative to the literature that is based on manipulating a black-box CSF. The current approach instead bases design on the observables.

Finally, the stochastic performance setting presents new research questions. For instance, what is the optimal design of contests with multiple rounds? Or the optimal design of a contest between teams where only the aggregate performances of each team is observed? Such questions have been considered in the literature before but not by starting from the stochastic-performance foundation. Thus, more research into the full implications of contests with stochastic performance is needed.

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Appendix A: Omitted proofs

Proof of Proposition 1. The proof characterizes the set of implementable a_i and proves the assertion in Proposition 1. Let $\widehat{q}_i(a_i)$ denote the unique value of q_i for which $L_i(q_i|a_i) = 0$. Now fix some target action, a_i^t, that the designer may wish to implement. As explained at the beginning of Section 3.1, when evaluated at $a_i = a_i^t$ (3) is maximized with a threshold assignment rule that has the property that

$$P_i(q_i, \mathbf{q}_{-i}) = \begin{cases} 1 & \text{if } q_i \geq \widehat{q}_i(a_i^t) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

The threshold rule in (19) is useful because by construction it maximizes the first derivative in (3) when evaluated at $a_i = a_i^t$. Hence, if the threshold rule leads (3) to take a negative value, then there is no assignment rule that can feasible satisfy the first-order condition. Then, the target action a_i^t simply cannot be implemented. Hence, it is necessary for implementability that (3) is non-negative at a_i^t when the threshold rule is used. This is a sufficient condition as well. To see this, consider a threshold rule with threshold $q_i = \underline{q}_i$. Then, the agent never wins, regardless of his performance. Hence, (3) is strictly negative at $a_i = a_i^t$. By continuity, there must then exist some threshold between $\underline{q}_i$ and $\widehat{q}_i(a_i^t)$ for which (3) is exactly zero when evaluated at $a_i = a_i^t$. Since this threshold rule is monotonic, the agent's expected utility is concave by the CDFC and the first-order condition is thus sufficient.

More precisely, given (19), agent i's expected utility from some action a_i is

$$v_i(1 - G_i(\widehat{q}_i(a_i^t)|a_i)) - a_i. \tag{20}$$

Hence, following the above argument, a_i^t is implementable if and only if

$$-\frac{\partial G_i(\widehat{q}_i(a_i^t)|a_i)}{\partial a_i}_{|a_i = a_i^t} \geq \frac{1}{v_i}. \tag{21}$$

The MLRP implies that the left-hand side is strictly positive.

Moreover, the $P_i(q_i, \mathbf{q}_{-i})$ function that implements a_i is (essentially) unique if and only if (21) is binding. First, $P_i(q_i, \mathbf{q}_{-i})$ is not unique when (21) is slack.

It has already been established that there is a threshold rule that implements such an a_i. However, by similar reasoning, there is another threshold rule with threshold above $\widehat{q}_i(a_i^t)$ that satisfies the first-order condition. For any action for which (21) binds, the assignment rule is essentially unique in its description of $P_i(q_i, \mathbf{q}_{-i})$ because the threshold rule maximizes (3). Thus, any assignment rule that differs on a set of performances profiles of positive measure would fail to satisfy the agent's first-order condition. This last part proves the proposition. ■

Proof of Proposition 2. The task is to identify action profiles that are group-symmetric and on the frontier of the feasible set. The corners are by definition where agents in group i take action $\underline{a}_i^s$ and agents in group j take action $\overline{a}_j^s$, $i \neq j$, and $i, j = 1, 2$. This proposition describes the rest of the frontier. Here, both groups must take actions strictly higher than $\underline{a}_i^s$, $i = 1, 2$. Otherwise, the other group j can be induced to take action $\overline{a}_j^s$, but this describes either a corner (if $a_i = \underline{a}_i^s$) or a point on the boundary that is not on the frontier (if $a_i < \underline{a}_i^s$). Hence, the action in group i is in $(\underline{a}_i^s, \overline{a}_i^s)$, $i = 1, 2$. Since actions are interior, incentive compatibility necessitates that the agents' first-order conditions are satisfied. The idea is to use the first-order approach by assuming (and then verifying) that the first-order conditions are also sufficient.

Ignoring group-symmetry to start, any action profile $\mathbf{a} = (a_j, \mathbf{a}_{-j})$ that is on the frontier must have the property that a_j is maximized given $\mathbf{a}_{-j}$. For a fixed j and $\mathbf{a}_{-j}$, the assignment rule must therefore solve

$$\max_{a_j, \{P_\omega(\mathbf{q})\}_{\omega \in \Omega, \mathbf{q} \in Q}} a_j \tag{22}$$

$$st \quad \frac{\partial U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i} = 0, \quad \text{for all } i \in N$$

$$P_\omega(\mathbf{q}) \geq 0, \quad \text{for all } \mathbf{q} \in Q \text{ and all } \omega \in \Omega$$

$$\sum\nolimits_{\omega \in \Omega} P_\omega(\mathbf{q}) = 1, \quad \text{for all } \mathbf{q} \in Q,$$

where $Q = \times_{i \in N}[\underline{q}_i, \overline{q}_i]$. Combining (1) and (3) means that the first set of constraints can be written

$$\int \left(\sum\nolimits_{\{\omega \in \Omega | i \in \omega\}} P_\omega(\mathbf{q}) v_i L_i(q_i|a_i) \right) \prod\nolimits_{k \in N} g_k(q_k|a_k) d\mathbf{q} - 1 = 0 \text{ for all } i \in N$$

or

$$\mathbb{E}\left[\sum\nolimits_{\{\omega\in\Omega|i\in\omega\}} P_\omega(\mathbf{q})v_i L_i(q_i|a_i)|\mathbf{a}\right]-1=0.$$

It is convenient to write the second and third sets of constraints as

$$\begin{aligned} P_\omega(\mathbf{q})\prod\nolimits_{k\in N} g_k(q_k|a_k) &\geq 0, \quad \text{for all } \mathbf{q}\in Q \text{ and all } \omega\in\Omega \\ \left(\sum\nolimits_{\omega\in\Omega} P_\omega(\mathbf{q})-1\right)\prod\nolimits_{k\in N} g_k(q_k|a_k) &= 0, \quad \text{for all } \mathbf{q}\in Q. \end{aligned}$$

Let $\{\mu_i\}_{i\in N}$ denote the multipliers to the first set of constraints and$\{\lambda_\omega(\mathbf{q})\}_{\omega\in\Omega,\ \mathbf{q}\in Q}$ and $\{\eta(\mathbf{q})\}_{\mathbf{q}\in Q}$ the multipliers to the second and third set of constraints, respectively. The Lagrangian can then be written as

$$a_j+\mathbb{E}\left[\sum_{i\in N}\sum_{\{\omega\in\Omega|i\in\omega\}} P_\omega(\mathbf{q})\mu_i v_i L_i(q_i|a_i)+\sum_{\omega\in\Omega}\lambda_\omega(\mathbf{q})P_\omega(\mathbf{q})+\eta(\mathbf{q})\left(\sum_{\omega\in\Omega}P_\omega(\mathbf{q})-1\right)|\mathbf{a}\right]-\sum_{i\in N}\mu_i$$

For a given assignment ω and a given performance profile $\mathbf{q}$, the first-order condition with respect to $P_\omega(\mathbf{q})$ is

$$\sum_{i\in\omega}\mu_i v_i L_i(q_i|a_i)+\lambda_\omega(\mathbf{q})=-\eta(\mathbf{q}),$$

where the right hand side is independent of ω. Hence, $\lambda_\omega(\mathbf{q})$ is smallest for the assignment ω which maximizes the first term on the left hand side. Since $\lambda_\omega(\mathbf{q})\geq 0$, this means that $\lambda_\omega(\mathbf{q})>0$ for all ω that do not maximize this first term. Thus, $P_\omega(\mathbf{q})=0$ for such assignments. Hence, feasibility dictates that

$$\sum\nolimits_{\omega\in\Omega(\mathbf{q},\boldsymbol{\mu}|\mathbf{a})} P_\omega(\mathbf{q})=1$$

where

$$\Omega(\mathbf{q},\boldsymbol{\mu}|\mathbf{a})=\{\omega\in\Omega|\sum\nolimits_{i\in\omega}\mu_i v_i L_i(q_i|a_i)\geq\sum\nolimits_{i\in\omega'}\mu_i v_i L_i(q_i|a_i) \text{ for all } \omega'\in\Omega\}.$$

Next, it is necessary to sign $\{\mu_i\}_{i\in N}$. To begin, since agent i is incentivized to take a positive action he must win a prize with strictly positive probability. Then, it is easy to rule out that $\mu_i<0$. In this case, by the MLRP, agent i's score diminishes when q_i increases, meaning that any assignment ω that he is a member of gets a lower aggregate score. Thus, any such assignment is less likely to be implemented. Stated differently, $P_i(q_i, \mathbf{q}_{-i})$ is decreasing in q_i if $\mu_i < 0$. This violates the incentive constraint in the maximization problem as the agent then has an incentive to deviate downwards. Hence, $\mu_i \geq 0$. The difficulty is in ruling out that $\mu_i = 0$. To this end, it is useful to consider the first-order condition for a_j in (22), which is

$$1 + \sum_{i \in N} \mu_i \frac{\partial^2 U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i \partial a_j} = 0. \tag{23}$$

It follows that μ_i cannot be zero for all $i \in N$. In other words, there is some agent $i \in N$ with $\mu_i > 0$. The aim is to show that $\mu_i > 0$ for all $i \in N$, or to rule out that $\mu_i = 0$ for any $i \in N$. Now, there is a problem like (22) for any $j \in N$. The equilibrium assignment rule must solve all these problems, or the action profile would not be on the frontier. Thus, regardless of which $j \in N$ is considered in (22), the same μ_i multipliers must solve the problem. By extension, (23) holds for all $j \in N$.

Now assume by contradiction that $\mu_j = 0$ for some agent $j \in N$. Consider how this latter agent j interacts with any agent i for which $\mu_i > 0$. Since $\mu_j = 0$, agent j's score is $\mu_j v_j L_j(q_j|a_j) = 0$ regardless of q_j. Thus, $\Omega(\mathbf{q}, \boldsymbol{\mu}|\mathbf{a})$ is independent of q_j. Therefore, q_j does not matter from agent i's point of view unless possibly if there are distinct assignments ω and ω' in $\Omega(\mathbf{q}, \boldsymbol{\mu}|\mathbf{a})$ such that agent i is a member of ω but not ω', in which case the value of q_j could be used as a tie-breaker to determine whether agent i receives a prize or not. However, this is a probability zero event. The reason is that $\mu_i > 0$ means that agent i's score is strictly increasing in q_i. Therefore, given $\mathbf{q}_{-i}$, the aggregate score of any assignment of which agent i is a member is strictly increasing in q_i.

Thus, q_j does not impact agent i. A marginal increase in a_j changes the distribution of q_j, but this is irrelevant to agent i. Hence, $\mu_i \frac{\partial^2 U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i \partial a_j} = 0$ if $\mu_i > 0$ and $\mu_j = 0$. Thus, all the term under the summation sign in (23) are zero, which means that (23) is violated. It follows that $\mu_j > 0$ for all $j \in N$.

Since the multipliers are positive, any agent obtains a strictly higher score the higher his performance is, by the MLRP. Thus, the probability that he is assigned a prize increases. In other words, $P_i(q_i, \mathbf{q}_{-i})$ is monotonic in q_i and the CDFC now implies that the agent's problem is concave. Hence, the first-order condition is sufficient. That is, the first-order approach is valid.

Thus far, group symmetry has not been invoked. Hence, the proof demonstrates a principle that extends to contests with more groups or in which group symmetry is not imposed. However, as discussed in Section 6.1, there are other complications in that case. Thus, the remainder of the proof makes use of the assumption that there are exactly two groups and that group symmetry is imposed. Assume that agents in group 1 must all be induced to take the same action, a_1. This means that two distinct member of group 1 must have multipliers that take the same value. Otherwise, the agent with the higher μ_i wins more often when his likelihood-ratio is positive and less often when his likelihood-ratio is negative than the agent with the lower multiplier does. However, this means that the former has stronger incentives than the latter on the margin, starting at the common action a_1. This violates the incentive constraint of at least one of the agents. Therefore, the multipliers must be group symmetric. ■

Proof of Proposition 3. The proof follows the same steps as the proof of Proposition 2, but modified to account for the designer's more general preferences. Given assignment rule $\mathbf{P}$ and action profile $\mathbf{a}$, the designer's expected utility is

$$\begin{aligned} U_v(\mathbf{a}|\mathbf{P}) &= \mathbb{E}[\textstyle\sum_{\omega\in\Omega} \pi_\omega(\mathbf{q}, \mathbf{a}) P_\omega(\mathbf{q})|\mathbf{a}] \\ &= U_0(\mathbf{a}) + \mathbb{E}[\textstyle\sum_{\omega\in\Omega} v_\omega(\mathbf{a}) P_\omega(\mathbf{q})|\mathbf{a}] \\ &= \mathbb{E}[\textstyle\sum_{\omega\in\Omega} \left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right) P_\omega(\mathbf{q})|\mathbf{a}], \end{aligned}$$

where $U_0(\mathbf{a})$ is again the expected value of $\pi(\mathbf{q}, \mathbf{a})$ and where the last equality follows from the fact that probabilities sum to one for all performance profiles. The objective is to maximize $U_v(\mathbf{a}|\mathbf{P})$ subject to the same feasibility constraints as in the proof of Proposition 2. The same arguments then establish that the score of any assignment is

$$\pi_\omega(\mathbf{q}, \mathbf{a}) + \textstyle\sum_{i\in\omega} \mu_i v_i L_i(q_i|a_i),$$

but since $\pi(\mathbf{q},\mathbf{a})$ cancel out, scores can instead be computed as

$$\upsilon_\omega(\mathbf{a})+\sum\nolimits_{i\in\omega}\mu_i v_i L_i(q_i|a_i).$$

The optimal assignment rule must assign probability one among the assignments with the highest scores. This produces the rule in Proposition 3.

To sign the multipliers, consider the first-order condition for a_j in the maximization problem,

$$\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j}+\sum\nolimits_{i\in N}\mu_i\frac{\partial^2 U_i(a_i,\mathbf{a}_{-i}|\mathbf{P}^*)}{\partial a_i\partial a_j}=0, \tag{24}$$

where $\mathbf{P}^*$ is an optimal assignment rule. The first step is to show that $\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j}>0$ if $\mu_j=0$. To this end, assume that $\mu_j=0$ and note that q_j as a consequence does not impact the score of any assignment. Thus, let $\Omega_{\mathbf{q}_{-j}}$ denote the set of assignments with the highest scores, given $\mathbf{q}_{-j}$. This may have several elements, but even in this case the value of $\upsilon_\omega(\mathbf{a})$ is the same for all $\omega\in\Omega_{\mathbf{q}_{-j}}$ for almost all $\mathbf{q}_{-j}$.[25] Now write $U_v(\mathbf{a}|\mathbf{P}^*)$ as

$$U_v(\mathbf{a}|\mathbf{P}^*)=\int\left(\int\sum\nolimits_{\omega\in\Omega_{\mathbf{q}_{-j}}}\left(U_0(\mathbf{a})+\upsilon_\omega(\mathbf{a})\right)P^*_\omega(\mathbf{q})g_j(q_j|a_j)dq_j\right)\prod\nolimits_{i\neq j}g_i(q_i|a_i)d\mathbf{q}_{-j}.$$

Then, $\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j}$ is determined by the derivative of the inner integral, which for a fixed $\mathbf{q}_{-j}$ is

$$\begin{aligned}\int\left(\sum\nolimits_{\omega\in\Omega_{\mathbf{q}_{-j}}}\frac{\partial\left(U_0(\mathbf{a})+\upsilon_\omega(\mathbf{a})\right)}{\partial a_j}P^*_\omega(\mathbf{q})g_j(q_j|a_j)dq_j\right)&\\ +\int\left(\sum\nolimits_{\omega\in\Omega_{\mathbf{q}_{-j}}}\left(U_0(\mathbf{a})+\upsilon_\omega(\mathbf{a})\right)P^*_\omega(\mathbf{q})L_j(q_j|a_j)g_j(q_j|a_j)dq_j\right)&\end{aligned}$$

By assumption, $a_j>0$. This necessitates that agent j has a strictly positive probability of winning under $\mathbf{P}^*$. Hence, the first line is strictly positive for a set

[25] For example, if $\mu_i=0$ and $a_i=a$ for all agents, then any assignment that allocates all prizes yields the same score and the same value of $\upsilon_\omega(\mathbf{a})$ if $\upsilon_\omega(\mathbf{a})=\sum_{i\in\omega}a_i$. However, more generally, two assignments ω and ω' with $\upsilon_\omega(\mathbf{a})\neq\upsilon_{\omega'}(\mathbf{a})$ obtain the same score only if $\sum_{i\in\omega}\mu_i v_i L_i(q_i|a_i)-\sum_{i\in\omega'}\mu_i v_i L_i(q_i|a_i)=\upsilon_{\omega'}(\mathbf{a})-\upsilon_\omega(\mathbf{a})\neq 0$, which occurs with probability zero.

of $\mathbf{q}_{-j}$ of positive measure, by (6). Turning to the second line, for almost all $\mathbf{q}_{-j}$, the value of $U_0(\mathbf{a}) + \upsilon_\omega(\mathbf{a})$ across $\Omega_{\mathbf{q}_{-j}}$ is, as explained above, unique. Hence, the fact that a change in a_j changes the distribution of q_j and with it potentially the choice of assignment in $\Omega_{\mathbf{q}_{-j}}$ has no impact almost always. Hence, the expectation of the first line is strictly positive, while the expectation of the second line is zero. Therefore, $\frac{\partial U_\upsilon(\mathbf{a}|\mathbf{P}^*)}{\partial a_j} > 0$ if $\mu_j = 0$. Thus, the first term in (24) is strictly positive and the same arguments as in the proof of Proposition 2 can then be used again to complete the proof of Proposition 3. ■

Proof of Proposition 4. This is a corollary of the more general results in Appendix B, specifically Corollary 1, Proposition 10, and Proposition 13. ■

Proof of Proposition 5. From (16),

$$a_1 + a_2 = r_1 \upsilon_1 F\left(\frac{1}{c}\right) + r_2 \upsilon_2 F(c).$$

Hence, it is possible to view the problem of maximizing $a_1 + a_2$ as a maximization problem in c. By assumption, $r_2 \upsilon_2 \geq r_1 \upsilon_1$. Hence, maximization requires that $F(c) \geq F\left(\frac{1}{c}\right)$. Since $F(c)$ is strictly increasing, this means that $c \geq 1$. Regardless of r, the first-order condition to this maximization problem implies that

$$k = \frac{F'\left(\frac{1}{c}\right)}{c^2 F'(c)},$$

where $k = \frac{r_2 \upsilon_2}{r_1 \upsilon_1} \geq 1$ is exogenous. Given $c \geq 1$, it can be verified that the right-hand side simplifies to exactly c. Hence, $c = k \geq 1$ is necessary. Using this and the definition of k in (16) yields (a_1, a_2) as stated in the proposition. Then, (17) follows. ■

Proof of Proposition 6. This follows from Proposition 9 and Proposition 12 in Appendix B when the CSFs are evaluated at $\mathbf{a} = \mathbf{a}^*$. ■

Proof of Proposition 7. Agent i's final score is below s_i if and only if both s_i^B and s_i^{AUX} are below s_i. First, $s_i^B \leq s_i$ when $q_i \leq H^{-1}\left(s_i^{b_i}\right)$, the probability of which is $H(q_i)^{f_i(a_i)} = s_i^{b_i f_i(a_i)}$. Second, the probability that $s_i^{AUX} \leq s_i$ is $s_i^{\delta_i}$. Hence, the probability that the final score is below s_i is $s_i^{b_i f_i(a_i)} s_i^{\delta_i} = s_i^{b_i f_i(a_i) + \delta_i}$. It is as if agent i draws $b_i f_i(a_i) + \delta_i$ "ideas" from a uniform distribution. Similarly, the designer draws z "ideas" from a uniform distribution. Since each "idea" is equally likely to be the best, the ex ante probability that agent i wins is (18). ■

Appendix B: Best-shot vs. exponential-noise

This appendix studies the best-shot and exponential-noise models with one prize in more detail. In the best-shot model, it is possible to handle more agents and more general impact functions.

The best-shot model

Section 6.1 noted the complications that may arise in contests with many agents. Essentially, the frontier of the feasible set has many "parts." To illustrate, the first results focus on the special case where all agents are in one large "block." Then, the assignment rule is described as in Proposition 2, but allowing for more agents. That is, agent i earns a score of $\mu_i v_i L_i(q_i|a_i)$ and the agent with the highest score wins. Since there is only one block, $\mu_i \in (0, \infty)$ for all $i \in N$. Using the notation from Section 5.1, this means that $\tau_i \in (0, \infty)$. From (13), τ_i is in a sense a measure of how favorable the contest is to agent i, as demonstrated in the following result.

Proposition 8 *Consider the best-shot model with $f_j'(a_j) > 0 \geq f_j''(a_j)$ for all $j \in N$. Fix an action profile $\mathbf{a}^*$ on the frontier of the feasible set in which all agents are active and in the same block. Then, agent i's ex ante equilibrium winning probability exceeds that of agent j if and only if $\tau_i > \tau_j$.*

Proof. Note that if agents i and j perform equally well given what is expected of them – i.e. they perform at the same quantiles, or $G_i(q_i|a_i^*) = G_j(q_j|a_j^*)$ – then agent i's score beats agent j's score if $\tau_i > \tau_j$ and the likelihood-ratios are positive. However, agent's j's score is higher if the likelihood-ratios are negative.

Consequently, the result is trivial if rationing is allowed. Then, only positive likelihood-ratios have a chance of winning. Recall that agents i and j have positive likelihood-ratios with the same probability, specifically $1 - e^{-1}$. Given a performance at any fixed quantile above e^{-1}, such that $G_i(q_i|a_i^*) = G_j(q_j|a_j^*) \geq e^{-1}$, agent i outscores agent j if and only if $\tau_i > \tau_j$. Since quantiles are distributed the same way (uniformly) for both agents, it now follows that agent i wins with a higher probability in equilibrium if and only if $\tau_i > \tau_j$.

If rationing is ruled out, then performance with negative likelihood-ratio come into play. Given τ_i, agent i's score is in equilibrium distributed according to

$$K_i(s_i|\tau_i) = e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i]$$

with density

$$k_i(s_i|\tau_i) = \frac{1}{\tau_i} e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i].$$

Without loss of generality, arrange agents in ascending order based on their τ_i, with $\tau_1 \leq \tau_2 \leq ...\tau_N$. Let $\tau_0 = -\infty$. A score above τ_j automatically beats agent j. Hence, agent i's equilibrium winning probability can then be written as

$$\begin{aligned} P_i^*(\tau_1, \tau_2, ..., \tau_N) &= \int_{\tau_0}^{\tau_1} \left(\prod_{j\geq 1, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i) ds \\ &\quad + \int_{\tau_1}^{\tau_2} \left(\prod_{j\geq 2, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i) ds \\ &\quad +... + \int_{\tau_{i-1}}^{\tau_i} \left(\prod_{j\geq i, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i) ds \\ &= \frac{1}{\tau_i} \sum_{m=1}^{i} \alpha_m, \end{aligned}$$

where

$$\begin{aligned} \alpha_m &= \int_{\tau_{m-1}}^{\tau_m} e^{\sum_{j\geq m}\left(\frac{s}{\tau_j}-1\right)} ds \\ &= \frac{1}{\sum_{j\geq m} \frac{1}{\tau_j}} \left(e^{\sum_{j\geq m}\left(\frac{\tau_m}{\tau_j}-1\right)} - e^{\sum_{j\geq m}\left(\frac{\tau_{m-1}}{\tau_j}-1\right)}\right). \end{aligned}$$

Going forward, for $i = 2, ..., n$, it is useful to compare

$$\alpha_i = \frac{1}{\sum_{j\geq i} \frac{1}{\tau_j}} \left(e^{\sum_{j\geq i}\left(\frac{\tau_i}{\tau_j}-1\right)} - e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)}\right)$$

and

$$\begin{aligned}\sum_{m=1}^{i-1}\alpha_m &\leq \int_{\tau_0}^{\tau_{i-1}} e^{\sum_{j\geq i-1}\left(\frac{s}{\tau_j}-1\right)}ds.\\ &= \frac{1}{\sum_{j\geq i-1}\frac{1}{\tau_j}}e^{\sum_{j\geq i-1}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)}\\ &= \frac{1}{\sum_{j\geq i-1}\frac{1}{\tau_j}}e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)}.\end{aligned}$$

Then, note that for $i = 2, ..., n$,

$$\begin{aligned}P_i^* - P_{i-1}^* &= \frac{1}{\tau_i}\sum_{m=1}^{i}\alpha_m - \frac{1}{\tau_{i-1}}\sum_{m=1}^{i-1}\alpha_m\\ &= \frac{1}{\tau_i}\alpha_i - \left(\frac{1}{\tau_{i-1}}-\frac{1}{\tau_i}\right)\sum_{m=1}^{i-1}\alpha_m\\ &\geq \frac{1}{\tau_i}\frac{1}{\sum_{j\geq i}\frac{1}{\tau_j}}\left(e^{\sum_{j\geq i}\left(\frac{\tau_i}{\tau_j}-1\right)} - e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)}\right)\\ &\quad -\left(\frac{1}{\tau_{i-1}}-\frac{1}{\tau_i}\right)\frac{1}{\sum_{j\geq i-1}\frac{1}{\tau_j}}e^{\sum_{j\geq i}\left(\frac{\tau_{i-1}}{\tau_j}-1\right)},\end{aligned}$$

and where, defining $x_i = \sum_{j\geq i}\frac{1}{\tau_j}$, the latter is proportional to

$$\begin{aligned}\Delta_i &= (1+\tau_{i-1}x_i)\left(e^{\tau_i x_i} - e^{\tau_{i-1}x_i}\right) - (\tau_i - \tau_{i-1})\,x_i e^{\tau_{i-1}x_i}\\ &= (1+\tau_{i-1}x_i)\,e^{\tau_i x_i} - (1+\tau_i x_i)\,e^{\tau_{i-1}x_i} > 0\end{aligned}$$

when $\tau_i > \tau_{i-1}$. Hence, it now follows that winning probabilities are arranged in the same order as the τ_i's. ■

Given a vector $\boldsymbol{\tau}$ that lists all τ_i's, it is in principle possible to derive the CSF – the probability that agent i wins for any given action profile $\mathbf{a}$ – by integrating out the uncertainty over performance, i.e. by calculating

$$\int\left(\int P_i(q_i,\mathbf{q}_{-i})g_i(q_i|a_i)dq_i\right)\prod_{j\neq i}g_j(q_j|a_j)d\mathbf{q}_{-i}.$$

In the best-shot model, however, a more direct argument is also possible. This is illustrated in the proof of the next proposition, under the assumption that rationing is ruled out and that all agents are active. In this case, negative scores have a chance of winning.

Proposition 9 *Under the assumptions in Proposition 8, if* $\mathbf{a}^*$ *is the equilibrium action profile and* $\tau_i \leq \tau_j$ *for all* $j \in N$*, then agent* i *wins with probability*

$$\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau}) = \left(\prod_{j \in N \setminus \{i\}} e^{\frac{\left(\tau_i - \tau_j\right) f_j(a_j)}{\tau_j f_j(a_j^*)}}\right) \frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j \in N} \frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \tag{25}$$

for any action profile $\mathbf{a}$ *with* $a_i > 0$*.*

Proof. To start, note that the distribution of agent i's score is

$$S_i(s|a_i) = \left(e^{s-\tau_i}\right)^{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}, \quad s \in (-\infty, \tau_i]$$

when he takes action a_i rather than a_i^*. It is as if he draws $\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}$ scores from the distribution $e^{s-\tau_i}$, but only the best score is counted. The range of scores depends on the identity of the agent, with τ_i describing the highest possible score that agent i can achieve. Assume agent i is the agent with the lowest τ, or $\tau_i \leq \tau_j$. Then, in order for agent i to win it is necessary that all other agents score below τ_i, the probability of which is

$$\left(\prod_{j \in N \setminus \{i\}} e^{\frac{\left(\tau_i - \tau_j\right) f_j(a_j)}{\tau_j f_j(a_j^*)}}\right). \tag{26}$$

Given this event, however, the conditional distribution of agent j's score is

$$\frac{S_j(s|a_j)}{S_j(\tau_i|a_j)} = \left(e^{s-\tau_i}\right)^{\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \quad s \in (-\infty, \tau_i].$$

Hence, it is as if all agents draw scores from the *same* distribution, $e^{s-\tau_i}$. Since each draw therefore has an equal chance of winning, the conditional probability that agent i wins is

$$\frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j\in N}\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}. \tag{27}$$

Combining (26) and (26) yields the CSF in the proposition. ■

As a consistency check, note that if $\tau_i = \tau_j$ for all $j \in N$ then $\widehat{p}_i(\mathbf{a}^*|\boldsymbol{\tau}) = \frac{1}{n}$ and all agents win with equal probability in equilibrium. Note that the first term in (25) depends on the action profile, for reasons that are carefully explained in the proof of the proposition. Due to this distortion, (25) is *not* a lottery CSF (except in the special case where $\tau_i = \tau_j$ for all $j \in N$).

The highest possible implementable action of agent i, $\overline{a}_i$, can be characterized succinctly in the best-shot model. This follows from the proof of Proposition 1.

Corollary 1 *Assume G_i takes the form in (9) with $f_i(0) = 0$ and $f_i'(\cdot) > 0 \geq f_i''(\cdot)$. Then, any action no greater than the unique solution $\overline{a}_i$ to*

$$\frac{f_i(\overline{a}_i)}{f_i'(\overline{a}_i)} = \frac{v_i}{e} \tag{28}$$

can be implemented by appropriately designing the assignment rule.

Proof. In the best-shot model, where $\widehat{q}_i(a_i^t) = H^{-1}\left(e^{-\frac{1}{f_i(a_i^t)}}\right)$ or $H(\widehat{q}_i(a_i)) = e^{-\frac{1}{f_i(a_i^t)}}$, (20) is

$$\overline{U}_i(a_i) = v_i\left(1 - e^{-\frac{f_i(a_i)}{f_i(a_i^t)}}\right) - a_i$$

and (21) simplifies to

$$\frac{f_i'(a_i^t)}{f_i(a_i^t)} \geq \frac{e}{v_i}.$$

By concavity, the left hand side is decreasing. Hence, the condition is satisfied if and only a_i^t is no greater than the solution to (15). By Proposition 1, it is then possible to implement the action. ■

The assumption that $f_i(0) = 0$ ensures that (15) has a solution.[26] In the $n = 2$ agent case, it is possible to characterize the frontier of the feasible set explicitly.

[26] In contrast, if $\frac{f_i(0)}{f'(0)}$ is large relative to v_i, then the agent's productivity at $a_i = 0$ is already large relative to his marginal productivity and to his valuation of the prize. In this case, it is impossible to incentivize the agent to actively participate.

Consider the case where rationing is ruled out first.

Proposition 10 *Consider the two-agent best-shot model with $f_i(0) = 0$, $f_i'(a_i) > 0 \geq f_i''(a_i)$. When rationing is ruled out, the frontier of the feasible set contains the corners $(\overline{a}_1, 0)$ and $(0, \overline{a}_2)$. Any interior action profile $\mathbf{a}^* = (a_1^*, a_2^*)$ is on the frontier if and only if*

$$\frac{f_1(a_1^*)}{f_1'(a_1^*)} = v_1 F\left(\frac{1}{c}\right) \text{ and } \frac{f_2(a_2^*)}{f_2'(a_2^*)} = v_2 F(c) \tag{29}$$

for some $c \in (0, \infty)$, where $c = \frac{\tau_2}{\tau_1}$ and

$$F(c) = \begin{cases} \frac{c}{(1+c)^2} e^{c-1} & \text{if } c \in (0,1) \\ \frac{c^2+c-1}{(1+c)^2} e^{\frac{1-c}{c}} & \text{if } c \geq 1 \end{cases}.$$

Here, $F(c)$ is strictly increasing in c and satisfies $F(1) = \frac{1}{4}$. Hence, a_2^* is increasing in c and a_1^* is decreasing in c.

Proof. Maximal actions were described in Corollary 1. To implement $\overline{a}_i$ using the threshold rule, the prize is dumped with agent j when agent i does not meet his threshold. Hence, agent j cannot influence the assignment and $a_j = 0$ is therefore the best response. The corner solutions are essentially obtained by letting $\tau_1 = 0$ or $\tau_2 = 0$. Thus, consider in the following $\tau_1, \tau_2 > 0$.

For interior actions and a fixed $\boldsymbol{\tau} = (\tau_1, \tau_2)$, the two first-order conditions must be solved. Assume that $\tau_i \leq \tau_j$. Then, regardless of his performance, agent i wins with a probability strictly less than one when $\tau_i < \tau_j$. He wins if $s_i(q_i) \geq s_j(q_j)$, which occurs if and only if q_i and q_j are such that

$$e^{\frac{\tau_i - \tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}} \geq G_j(q_j|a_j^*),$$

where the term on the right hand side is the equilibrium distribution of j's performance. Hence, the interim probability that agent i with performance q_i wins is $e^{\frac{\tau_i - \tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}$. With this in mind, agent i's first-order condition can be written

$$\int_{\underline{q}_i}^{\overline{q}_i} v_i L_i(q_i|a_i^*) \left(e^{\frac{\tau_i - \tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}} \right) g_i(q_i|a_i^*) dq_i - 1 = 0$$

or

$$\int_{\underline{q}_i}^{\overline{q}_i} \frac{\tau_i}{\mu_i}\left(1+\ln G_i(q_i|a_i^*)\right)\left(e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}\right) g_i(q_i|a_i^*)dq_i - 1 = 0.$$

Substituting the equilibrium quantiles of agent i's performance, $z = G_i(q_i|a_i^*)$ and $dz = g_i(q_i|a_i^*)dq_i$, yields

$$\begin{aligned} 1 &= \frac{\tau_i}{\mu_i} e^{\frac{\tau_i-\tau_j}{\tau_j}} \int_0^1 \left(1+\ln z\right) z^{\frac{\tau_i}{\tau_j}} dz \\ &= v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)} e^{\frac{\tau_i}{\tau_j}-1} \frac{\frac{\tau_i}{\tau_j}}{\left(\frac{\tau_i}{\tau_j}+1\right)^2}, \end{aligned}$$

which by concavity of $f_i(a_i)$ nails down a_i^*.

Assume now that $\tau_i > \tau_j$. In this case, agent i wins with probability one if his performance is high enough, or specifically if $q_i \geq \widetilde{q}_i$ where

$$e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(\widetilde{q}_i|a_i^*)^{\frac{\tau_i}{\tau_j}} = 1,$$

which implies that

$$G_i(\widetilde{q}_i|a_i^*) = e^{\frac{\tau_j-\tau_i}{\tau_i}}$$

and

$$1+\ln G_i(\widetilde{q}_i|a_i^*) = \frac{\tau_j}{\tau_i}.$$

Agent i's first order condition is now

$$\int_{\underline{q}_i}^{\widetilde{q}_i} v_i L_i(q_i|a_i^*)\left(e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}\right) g_i(q_i|a_i^*)dq_i + \int_{\widetilde{q}_i}^{\overline{q}_i} v_i L_i(q_i|a_i^*) g_i(q_i|a_i^*)dq_i - 1 = 0.$$

The same substitution as before yields

$$1 = e^{\frac{\tau_i-\tau_j}{\tau_j}} \int_0^{e^{\frac{\tau_j-\tau_i}{\tau_i}}} \frac{\tau_i}{\mu_i}\left(1+\ln z\right) z^{\frac{\tau_i}{\tau_j}} dz + \int_{e^{\frac{\tau_j-\tau_i}{\tau_i}}}^1 \frac{\tau_i}{\mu_i}\left(1+\ln z\right) dz$$

or

$$1 = v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)} e^{\frac{1}{\tau_i/\tau_j}-1} \frac{\left(\frac{\tau_i}{\tau_j}\right)^2 + \frac{\tau_i}{\tau_j} - 1}{\left(\frac{\tau_i}{\tau_j}+1\right)^2}.$$

As before, a_i^* is nailed down by concavity of $f_i(a_i)$.

Now let $j = 1$ and $i = 2$. The first-order condition for agent 2 can be written as

$$\frac{f_2(a_2^*)}{f_2'(a_2^*)} = v_2 F(c)$$

where $c = \frac{\tau_2}{\tau_1}$ and

$$F(c) = \begin{cases} \frac{c}{(1+c)^2} e^{c-1} & \text{if } c \in (0,1) \\ \frac{c^2+c-1}{(1+c)^2} e^{\frac{1-c}{c}} & \text{if } c \geq 1 \end{cases}.$$

Since $\frac{\tau_1}{\tau_2} = \frac{1}{c}$, it likewise follows that

$$\frac{f_1(a_1^*)}{f_1'(a_1^*)} = v_1 F\left(\frac{1}{c}\right).$$

Simple differentiation shows that $F(c)$ is strictly increasing and it is easy to verify that $F(1) = \frac{1}{4}$. The last part of the proposition then follows immediately. ■

The frontier of the feasible set is described in a similar fashion when rationing is allowed.

Proposition 11 *Consider the two-agent best-shot model with $f_i(0) = 0$, $f_i'(a_i) > 0 \geq f_i''(a_i)$. When rationing is allowed, the frontier of the feasible set contains the corners $(\overline{a}_1, \underline{a}_2)$ and $(\underline{a}_1, \overline{a}_2)$, where $\overline{a}_i$ and $\underline{a}_i$ solve*

$$\frac{f_i(\overline{a}_i)}{f_i'(\overline{a}_i)} = \frac{v_i}{e} \text{ and } \frac{f_i(\underline{a}_i)}{f_i'(\underline{a}_i)} = \frac{v_i}{e^2}, \quad i = 1, 2.$$

Any interior action profile $\mathbf{a}^* = (a_1^*, a_2^*)$ is on the frontier if and only if

$$\frac{f_1(a_1^*)}{f_1'(a_1^*)} = v_1 F_R\left(\frac{1}{c}\right) \text{ and } \frac{f_2(a_2^*)}{f_2'(a_2^*)} = v_2 F_R(c)$$

for some $c \in (0, \infty)$, where $c = \frac{\tau_2}{\tau_1}$ and

$$F_R(c) = \begin{cases} \frac{ce^{c-1}+e^{-2}}{(1+c)^2} & \text{if } c \in (0,1) \\ \frac{(c^2+c-1)e^{\frac{1-c}{c}}+e^{-2}}{(1+c)^2} & \text{if } c \geq 1 \end{cases}.$$

Here, $F(c)$ is strictly increasing in c and satisfies $F(1) = \frac{1}{4}$. Hence, a_2^* is increasing in c and a_1^* is decreasing in c.

Proof. For $c \in (0, \infty)$, the proof follows the same steps as in the proof of Proposition 10. The only difference is that agent i now has zero probability of winning if $q_i < \widehat{q}_i(a_i^*)$ or, using the same substitution as in Proposition 10, if $z \leq G_i(\widehat{q}_i(a_i^*)|a_i^*) = e^{-1}$. Hence, the lower bounds on the integral that are evaluated in the proof of Proposition 10 change. This produces $F_R(c)$ as stated in the last part of the proposition.

Finally, $\underline{a}_i$ can be obtained by letting $c \to 0$. Alternatively, the logic in Corollary 1 can be applied, but with the change that agent i wins only if $q_i \geq \widehat{q}_i(a_i^*)$ and his rival fails to meet his threshold, $\widehat{q}_j(a_j^*)$, which occurs with probability e^{-1}. It is for this reason that the right hand side in the equation for $\underline{a}_i$ is e^{-1} times its counterpart for $\overline{a}_i$, where $\overline{a}_i$ is described in Corollary 1. ■

The only difference between $F(c)$ and $F_R(c)$ is the presence of the e^{-2} term in the latter. Thus, $F_R(c) > F(c)$. Since $f_i(a_i)$ is concave, it follows, as expected, that the action profile for any given c is higher when rationing is allowed than when it is not.

Proposition 5 in the main text established that when (12) holds, the design that maximizes $a_1 + a_2$ without rationing involves $\frac{\tau_2}{\tau_1} = \frac{r_2 v_2}{r_1 v_1}$ (or $c = k$). This also turns out to be optimal when rationing is permitted. The proof of this follows the same steps as in the proof of Proposition 5 and is thus omitted.

To illustrate, assume that $v_1 = 1$, $v_2 = 2$, and $r_1 = r_2 = 1$. Figure 1 describes the frontier of the feasible set without rationing (the thin curve) and with rationing (the thick curve), respectively. The solid dot in the figure indicates the action profile that is implemented when $\frac{\tau_2}{\tau_1} = 1$. Then, the playing field is made level and both agents win with probability 0.5. Clearly, this point is not on a tangent line to a level curve for $a_1 + a_2$. Hence, this is not an optimal design if the objective is to maximize total effort. The open circles describe the optimal action profiles when rationing is or is not allowed.[27]

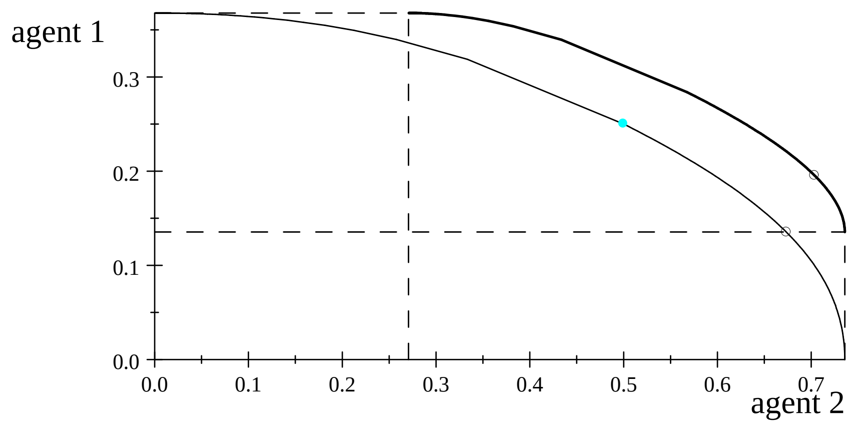

Figure 1: Comparing feasible sets and optimal action profiles.

Exponential noise

The exponential-noise model does not satisfy the CDFC. Thus, it is unclear at the outset whether first-order conditions are sufficient. This complication is ignored to start. Later, further restrictions on $f_i(a_i)$ will be imposed that ensure that the first-order conditions are in fact sufficient.

The counterpart to Propositions 8 and 9 is given next.

Proposition 12 *Consider the exponential-noise model with $n = 2$ agents and assume the first-order conditions are sufficient for incentive compatibility. For any action profile $\mathbf{a}^*$ along the frontier of the feasible set, agent i wins with a smaller probability than agent j if rationing is ruled out and $\tau_i > \tau_j$ but the opposite holds if rationing is permitted. If $\tau_i \geq \tau_j$ and rationing is ruled out,*

[27]Without rationing, $a_1 = \frac{2}{9}e^{-\frac{1}{2}} = 0.13478$ is optimal. In comparison, $\underline{a}_1 = \frac{1}{e^2} = 0.13534$. Hence, the former is marginally below the latter even though this is hard to see in the figure.

then agent i wins with probability

$$\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau}) = e^{\frac{\tau_j - \tau_i}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}} \frac{\frac{\tau_i f_i(a_i)}{f_i(a_i^*)}}{\frac{\tau_i f_i(a_i)}{f_i(a_i^*)} + \frac{\tau_j f_j(a_j)}{f_j(a_j^*)}}. \tag{30}$$

for any action profile $\mathbf{a}$ *with* $a_i > 0$*.*

Proof. Consider first an action profile along the frontier of the feasible set when rationing is permitted. Here, only positive likelihood-ratios have a chance of winning. From (14), the two agents are equally likely to have positive likelihood-ratios. For a fixed quantile, $G_i(q_i|a_i^*) = G_j(q_j|a_j^*)$, with a positive likelihood-ratio, agent i obtains a higher score than agent j since $\tau_i \geq \tau_j$. Hence, for any quantile, agent i is at least as likely to win as agent j. Hence, agent i is at least as likely as agent j to win ex ante.

Assume next that rationing is ruled out. Note that the probability that the likelihood-ratio is negative is substantial – it is $1 - e^{-1} = 0.632$ – and that in these cases agent i is hurt when $\tau_i \geq \tau_j$. The distribution of agent i's score is

$$S_i(s|a_i) = 1 - e^{-\frac{s+\tau_i}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}}, \quad s \in [-\tau_i, \infty)$$

when he takes action a_i rather than a_i^*. Assume now that $\tau_i \geq \tau_j$, which means that agent i has the lowest minimum score, or $-\tau_i \leq -\tau_j$. Then, agent i has a chance of winning only if his score is greater than $-\tau_j$, the probability of which is

$$1 - S_i(-\tau_j|a_i) = e^{\frac{\tau_j - \tau_i}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}}. \tag{31}$$

Conditional on agent i's score being at least $-\tau_j$, the probability that his score is at least s is

$$\frac{1 - S_i(s|a_i)}{1 - S_i(-\tau_j|a_i)} = e^{-\frac{s+\tau_j}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}},$$

which means that the conditional hazard rate is $\frac{1}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}$. Similarly, the hazard rate for agent j is $\frac{1}{\tau_j}\frac{f_j(a_j^*)}{f_j(a_j)}$. The defining characteristics here is that these hazard rates are constant for all $s \geq -\tau_j$.

Since there is only one competing agent, agent i wins if he is not the agent with the lowest score. Let the lowest score be denoted s' and assume that $s' \geq -\tau_j$.

Conditional on this s' value, agent i is thus the winner if it is agent j that obtained the score s'. Based on the hazard rates, this occurs with conditional probability

$$\frac{\frac{1}{\tau_j}\frac{f_j(a_j^*)}{f_j(a_j)}}{\frac{1}{\tau_i}\frac{f_i(a_i^*)}{f_i(a_i)}+\frac{1}{\tau_j}\frac{f_j(a_j^*)}{f_j(a_j)}}=\frac{\frac{\tau_i f_i(a_i)}{f_i(a_i^*)}}{\frac{\tau_i f_i(a_i)}{f_i(a_i^*)}+\frac{\tau_j f_j(a_j)}{f_j(a_j^*)}} \tag{32}$$

regardless of the exact value of s'. Combining (31) and (32) gives (11). At $\mathbf{a}=\mathbf{a}^*$, $\widehat{p}_i(\mathbf{a}^*|\boldsymbol{\tau})=e^{\frac{\tau_j-\tau_i}{\tau_i}}\frac{\tau_i}{\tau_i+\tau_j}$. This is strictly smaller than $\frac{1}{2}$ when $\tau_i>\tau_j$. ■

Note that $\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau})$ is not concave in a_i when $\tau_i>\tau_j$. It is for this reason that it is harder to check whether first-order conditions are sufficient or not. However, when they are sufficient, the implementable action profiles when rationing is ruled out are the same in the exponential noise model as in the best-shot model with $n=2$. The proof of the first part of the next result is instructive in explaining why this is. The second part provides a condition under which the first-order conditions are sufficient.

Proposition 13 *Consider the exponential-noise model with $n=2$ agents and assume that rationing is ruled out. If the first-order conditions are sufficient, then the frontier of the feasible set is described as in (29). The first-order conditions are sufficient when (12) holds.*

Proof. Consider interior action profiles. Agent i wins if his score, s_i, exceeds the score of agent j, s_j. At the equilibrium candidate action profile $\mathbf{a}^*$, this occurs if q_i and q_j satisfy

$$1-e^{\frac{\tau_i-\tau_j}{\tau_j}}(1-G_i(q_i|a_i^*))^{\frac{\tau_i}{\tau_j}}\geq G_j(q_j|a_j^*),$$

where the term on the right hand side is the "equilibrium" distribution of agent j's score. Hence, given q_i, agent i wins with probability

$$\max\{0,1-e^{\frac{\tau_i-\tau_j}{\tau_j}}(1-G_i(q_i|a_i^*))^{\frac{\tau_i}{\tau_j}}\}.$$

Note that if $\tau_i<\tau_j$ then agent i wins with strictly positive probability for all

q_i. In this case, agent i's first order condition at $\mathbf{a}^*$ can be written as

$$\int_{\underline{q}_i}^{\overline{q}_i} v_i L_i(q_i|a_i^*) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} (1 - G_i(q_i|a_i^*))^{\frac{\tau_i}{\tau_j}}\right) g_i(q_i|a_i^*) dq_i - 1 = 0$$

or

$$-\int_{\underline{q}_i}^{\overline{q}_i} \frac{\tau_i}{\mu_i} (1 + \ln (1 - G_i(q_i|a_i^*))) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} (1 - G_i(q_i|a_i^*))^{\frac{\tau_i}{\tau_j}}\right) g_i(q_i|a_i^*) dq_i - 1 = 0.$$

Substituting the equilibrium "survival quantiles" of agent i's performance, $z = 1 - G_i(q_i|a_i^*)$ and $dz = -g_i(q_i|a_i^*)dq_i$, yields

$$\begin{aligned}
1 &= \int_1^0 \frac{\tau_i}{\mu_i} (1 + \ln z)) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} z^{\frac{\tau_i}{\tau_j}}\right) dz \\
&= -\int_0^1 \frac{\tau_i}{\mu_i} (1 + \ln z)) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} z^{\frac{\tau_i}{\tau_j}}\right) dz \\
&= -\frac{\tau_i}{\mu_i} \int_0^1 (1 + \ln z)) \, dz + \frac{\tau_i}{\mu_i} e^{\frac{\tau_i - \tau_j}{\tau_j}} \int_0^1 (1 + \ln z)) \, z^{\frac{\tau_i}{\tau_j}} dz,
\end{aligned}$$

and since the first term in exactly zero, this first-order condition exactly coincides with its counterpart in the best-shot model.

Assume now that $\tau_i > \tau_j$. In this case, agent i wins with probability zero if his performance is low enough, or specifically if $q_i \leq \widehat{q}_i$ where

$$e^{\frac{\tau_i - \tau_j}{\tau_j}} (1 - G_i(\widehat{q}_i|a_i^*))^{\frac{\tau_i}{\tau_j}} = 1,$$

or

$$1 - G_i(\widehat{q}_i|a_i^*) = e^{\frac{\tau_j - \tau_i}{\tau_i}}$$

and

$$1 + \ln (1 - G_i(\widehat{q}_i|a_i^*)) = \frac{\tau_j}{\tau_i}.$$

Note how similar these expressions are to their counterparts in the best-shot model. Since agent i only win when his performance is high enough, his first order condition is

$$\int_{\widehat{q}_i}^{\overline{q}_i} v_i L_i(q_i|a_i^*) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} \left(1 - G_i(q_i|a_i^*)\right)^{\frac{\tau_i}{\tau_j}}\right) g_i(q_i|a_i^*) dq_i - 1 = 0.$$

The same substitution as before yields

$$\begin{aligned}
1 &= \int_{e^{\frac{\tau_j - \tau_i}{\tau_i}}}^{0} \frac{\tau_i}{\mu_i} (1 + \ln z)) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} z^{\frac{\tau_i}{\tau_j}}\right) dz \\
&= -\int_0^{e^{\frac{\tau_j - \tau_i}{\tau_i}}} \frac{\tau_i}{\mu_i} (1 + \ln z)) \left(1 - e^{\frac{\tau_i - \tau_j}{\tau_j}} z^{\frac{\tau_i}{\tau_j}}\right) dz \\
&= -\int_0^{e^{\frac{\tau_j - \tau_i}{\tau_i}}} \frac{\tau_i}{\mu_i} (1 + \ln z)) \, dz + e^{\frac{\tau_i - \tau_j}{\tau_j}} \int_0^{e^{\frac{\tau_j - \tau_i}{\tau_i}}} \frac{\tau_i}{\mu_i} (1 + \ln z)) \, z^{\frac{\tau_i}{\tau_j}} dz \\
&= \int_{e^{\frac{\tau_j - \tau_i}{\tau_i}}}^{1} \frac{\tau_i}{\mu_i} (1 + \ln z) \, dz + e^{\frac{\tau_i - \tau_j}{\tau_j}} \int_0^{e^{\frac{\tau_j - \tau_i}{\tau_i}}} \frac{\tau_i}{\mu_i} (1 + \ln z)) \, z^{\frac{\tau_i}{\tau_j}} dz
\end{aligned}$$

where the last equality comes from the fact that $\int_0^1 (1 + \ln z) \, dz = 0$. The first-order condition now takes the exact same form as in the best-shot model. This proves that the frontier of the feasible set is described as in (29). The corners are again obtained when $\tau_1 = 0$ or $\tau_2 = 0$.

Next, note that if $\tau_i \geq \tau_j$ then the CSF for agent i is described by $\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau})$ in (11). This means that $\widehat{p}_j(\mathbf{a}|\boldsymbol{\tau}) = 1 - \widehat{p}_i(\mathbf{a}|\boldsymbol{\tau})$. It is easily verified that this is concave in a_j whenever $f_j(a_j)$ is concave. Hence, the first-order condition is sufficient for agent j under fairly weak conditions. The problem is to verify that this is also the case for agent i. This is where the functional form in (12) comes into play. Given (12),

$$\widehat{p}_i(a_i, a_j^*|\boldsymbol{\tau}) = e^{\frac{1-\tau}{\tau}\left(\frac{a_i}{a_i^*}\right)^{-r_i}} \frac{\tau \left(\frac{a_i}{a_i^*}\right)^{r_i}}{\tau \left(\frac{a_i}{a_i^*}\right)^{r_i} + 1},$$

where $\tau = \frac{\tau_i}{\tau_j} \geq 1$. Note again that β_i is irrelevant. Differentiation and simplification shows that the sign of the second derivative of $\widehat{p}_i(a_i, a_j^*|\boldsymbol{\tau})$ with respect to

a_i is determined by the sign of

$$\Delta(x) = r_i(\tau-1)^2 + \left(1-\tau+r_i-3r_i\tau+2r_i\tau^2\right)\tau x \\ + \left(1-2\tau+2r_i-2r_i\tau+r_i\tau^2\right)\tau^2x^2-(1+r_i)\tau^4x^3$$

where $x = \left(\frac{a_i}{a_i^*}\right)^{r_i}$. Note that this is positive if x is small, but that it must be negative if x is sufficiently large. In other words, $\widehat{p}_i(a_i, a_j^*|\boldsymbol{\tau})$ is convex when a_i is small and concave when it is large.

Comparing the second and third coefficient, note that

$$\left(1-\tau+r_i-3r_i\tau+2r_i\tau^2\right)-\left(1-2\tau+2r_i-2r_i\tau+r_i\tau^2\right) = \tau(1-r_i)+\left(\tau^2-1\right)r_i > 0$$

for all $\tau > 1$ and all $r \in (0,1]$. This means that if the second coefficient is negative then the third coefficient must be negative as well. Thus, the four coefficients change signs exactly one. By Descartes' rule of sign, there is therefore exactly one positive root. Thus, the second derivative of $\widehat{p}_i(a_i, a_j^*|\boldsymbol{\tau})$ changes sign exactly once. It now follows that $\widehat{p}_i(a_i, a_j^*|\boldsymbol{\tau})$ is first-convex-then-concave in a_i. At $a_i = a_i^*$, or $x = 1$,

$$\Delta(1) = r_i + \tau(1-r_i) - 2\tau^3 - \tau^4 < 1 + \tau - 2\tau^3 - \tau^4 \leq 0.$$

Thus, payoff is locally concave at a_i^*. By the first-convex-then-concave shape of payoff, the only deviation from a_i^* that has to be ruled out is the deviation to $a_i = 0$. At $a_i = 0$, the probability of winning is zero, and payoff is therefore zero. Hence, there is no incentive to deviate from a_i^* if a_i^* gives non-negative expected payoff.

From (29) or (16), $a_i^* = r_i v_i F(\tau) = r_i v_i \frac{\tau^2+\tau-1}{(1+\tau)^2} e^{\frac{1-\tau}{\tau}}$, and expected payoff is then

$$\begin{aligned} v_i\widehat{p}_i^*(a_i, a_j^*|\boldsymbol{\tau}) &= v_i e^{\frac{1-\tau}{\tau}}\frac{\tau}{\tau+1} - r_i v_i \frac{\tau^2+\tau-1}{(1+\tau)^2} e^{\frac{1-\tau}{\tau}} \\ &= v_i e^{\frac{1-\tau}{\tau}}\left(\frac{\tau}{\tau+1} - r_i\frac{\tau^2+\tau-1}{(1+\tau)^2}\right) \\ &= v_i e^{\frac{1-\tau}{\tau}}\frac{\tau(1+\tau)(1-r_i)+r_i}{(\tau+1)^2} > 0, \end{aligned}$$

which confirms that there is no incentive to deviate from a_i^* to $a_i = 0$. Thus, a_i^* is the unique best response for agent i. This confirms that $\mathbf{a}^*$ is an equilibrium action profile given $\boldsymbol{\tau}$. $\blacksquare$