Contest Design with Stochastic Performance*

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Abstract

This paper studies optimal contest design in contests with noisy performance. Here, contest design is a team moral hazard problem that endogenizes the assignment rule that maps performance profiles into winning probabilities. The optimal design features endogenous standards for eligibility and the number of prizes that are awarded may be stochastic. Generally, one group of agents is identified as "first claimants" of prizes, contingent on performance exceeding a threshold of excellence. However, which group wins prizes more often depends on the designer's objective function and the performance technologies. Finally, the approach derives endogenous, microfounded, and fully optimal contest success functions.

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1 Introduction

A broad range of economic interactions are contest-like in nature. For the purposes of this paper, think of a contest as an environment in which rival agents take costly actions that influence the probability with which one of a number of fixed and identical prizes is won. Examples include rent-seeking, lobbying, college admission, innovation contests, promotion contests, sports, etc. The winners need not necessarily be the agents who took the most costly actions. In fact, actions are not even directly observable in most contests. Instead, prizes are typically awarded on the basis of some noisy signal, often interpretable as performance.

It is often natural to assume that prizes must be awarded to the agents with the "best" performance. There are several design issues in such unbiased contests, many of which have been explored elsewhere.[1] This paper in contrast considers the diametrically opposite case in which there is no obligation to award the prizes to the agents with the best signals. Utilizing a general and unifying model of contests, the paper thus concentrates on the optimal design of biased contests. Depending on the application, this may be implemented as preferential treatment, affirmative action, nepotism, or the like.

Consider a college admissions problem. Here, the student's high school GPA is observed. It is not uncommon in the US to differentiate between in-state and out-of-state students. For instance, in the UC system, an applicant from California who graduates in the top 9% of his high school cohort is guaranteed admission into one of the UC campuses. This guarantee is not extended to out-of-state students.[2] Similarly, many countries distinguish between domestic and foreign workers on the labor market. For instance, Canadian universities are obligated to hire a Canadian applicant if one exists that meets the bar. A foreign applicant can be hired only if no Canadian candidate meets the bar. In some countries, veterans are given similar preference. Finally, ethnic minorities are awarded bonus points in the very important National College Entrance Examination in China.

[1] Design issues include questions concerning what the optimal set of contestants is and how they are selected, entry fees, number and distribution of prizes, etc. For these and related questions, see e.g. Taylor (1995), Fullerton and McAfee (1999), Moldovanu and Sela (2001), Che and Gale (2003), Drugov and Ryvkin (2020), and Fang, Noe, and Strack (2020).

[2] Compare the in-state and out-of-state instructions (accessed October 6, 2020) at https://admission.universityofcalifornia.edu/admission-requirements/freshman-requirements/.

In principle, a high performing minority student can outscore even perfectly performing majority students. In all these examples, a member of the advantaged group does not have to worry about competition from outside his group, provided his own performance is high enough.

Another property of these examples is that agents must meet certain minimum requirements to have a chance of winning a prize. These "standards for eligibility" are part of the design, and are thus endogenous. Existing models typically abstract away from this design element but the current model is well suited to exploring the issue. This is pertinent in many kinds of contests and is relevant even if agents are symmetric ex ante. Consider research contests. The Ansari X Prize required a crewed spacecraft to be used twice in two weeks to enter space, a requirement that is clearly endogenous. In some contests, the requirements are very stringent indeed, and the prize may or may not be awarded as a result. Che and Gale (2003) mention the longitude rewards offered by the British government in 1714. These were predated by rewards offered by Spain and the Netherlands that were never awarded. The locomotive engine contest described by Che and Gale (2003) drew ten entrants but only one was able to complete the trial successfully. Similarly, in some promotion contests, it may happen that no agent is promoted if they all perform terribly. Finally, the standards for eligibility need not be symmetric. For instance, the entry requirements are more stringent for out-of-state students in the UC system.

The current paper views contests primarily as incentive schemes. Thus, contest design is intended to manipulated actions. In the model, the highest equilibrium actions are obtained with contest designs that have features like those mentioned in the previous two paragraphs. Thus, these features can be justified even without appealing to distributional considerations.

The following simple model is proposed. First, actions are not directly observable but a noisy and observable signal is produced by each agent. Typically, the noisy signal, q_i, can be thought of as the stochastic quality of agent i's performance. In a promotion contest among salespeople, a salesman's performance is his volume of sales. In innovation contests, a firm's performance is the quality of its innovation. In a competition for a merit-based scholarship or a seat at college, a student's performance includes his GPA to date. Similarly, a lobbyist's performance is how compelling he can make his agenda or proposal sound. The agent's action, a_i, impacts the distribution, $G_i(q_i|a_i)$, of his performance.[3]

The stochastic performance model nests popular models of contests. In Lazear and Rosen's (1981) rank-order tournament, the action shifts the location of the distribution function. Similarly, there are specifications of $G_i(q_i|a_i)$ for which the probability that agent i delivers the best performance exactly reduces to the lottery contest success function (CSF). Fullerton and McAfee's (1999) research tournament with a single prize is one such example.[4] Stated differently, Fullerton and McAfee (1999) provide microfoundations for the *unbiased* lottery CSF. However, no microfoundations have been offered for the *biased* lottery CSF that is often used in the literature. The resulting analysis can be criticized as being ad hoc or poorly founded. The current paper instead provides an internally consistent treatment of the optimal design of contests. Thus, the new approach has both methodological and practical implications.

The central idea is to view the problem as a kind of contracting or team moral hazard problem, with the distributions $G_i(q_i|a_i)$ as the primitives. Instead of offering wage schedules as in Holmström (1982), it is winning probabilities that are manipulated to incentivize effort. The designer's task is to construct and commit to an optimal "assignment rule" that maps performance profiles into winning probabilities, while keeping incentive compatibility constraints in mind.

The first part of the paper establishes that the general design principles are the same for a wide range of objective functions. The optimal assignment rule can be implemented by endowing each agent with a scoring function that is increasing but typically non-linear in performance. The higher the agent's score, the more likely it is that he wins a prize. Scoring rules are calibrated to provide the right incentives and may be identity-dependent.

[3] Recently, Bastani, Giebe, and Gürtler (2021) have independently proposed a similar model. However, their focus is on comparative statics in unbiased contests with a single prize. See also Ryvkin and Drugov (2020) and Drugov and Ryvkin (2020).

[4] For other justifications in this vein, see Hirschleifer and Riley (1992), Clark and Riis (1996), Baye and Hoppe (2003), and Jia (2008). Skapardas (1996) and Clark and Riis (1998) instead take an axiomatic approach to justifying the lottery CSF. Corchón and Dahm (2011) consider a designer who cannot commit but who is not an expected utility maximizer. These microfoundations are influential and emphasized in e.g. the surveys by Konrad (2009), Vojnonić (2015), Corchón and Serena (2018), and Fu and Wu (2019). For other surveys on biased contest design, see Mealem and Nitzan (2016) and Chowdhury, Esteve-González, and Mukherjee (2019).

Agents' likelihood-ratios play a key role in providing incentives because optimal scoring rules turn out to be piece-wise linear in likelihood-ratios. This is consistent with insights from the standard principal-agent model where the likelihood-ratio can be thought of as the incentive weight of any give wage. Negative likelihood-ratios should be punished if at all possible, which is where rationing comes in. Rationing is implemented by imposing endogenous standards for eligibility. These standards may or may not be met in equilibrium, which means that the number of prizes that are awarded is stochastic. Similarly, very large likelihood-ratios should be rewarded if at all possible, which is where policies like guaranteed admission and preferential hiring enter the picture. In fact, it is a robust feature of optimal contest design that a group of agents is singled out as "first claimants" of prizes. These agents are guaranteed to outscore agents outside their group as long as their performance exceeds a threshold of excellence.

These design principles are common to a host of objective functions and distribution functions. However, the details – in terms of which agents are enticed to work harder and which agents receive steeper scoring rules – are sensitive to the objective function. If the designer thinks of agents' actions as highly complementary, then a more balanced action profile is induced. When rationing is possible, this unambiguously leads to preferential treatment of "weaker" agents who end up being over-represented among the eventual winners. For instance, consider a government that thinks of effort in high-school as a predictor of future success and uses the rules that governs entry into college to incentivize such effort. If the government has Rawlsian (or maximin) preferences, the weaker group is given preferential treatment even though the government is not concerned about agents' identities per se. On the other hand, if actions are close to perfect substitutes, the designer may focus her attention on incentivizing "stronger" agents. The second part of the paper explores these issues in more detail when $G_i(q_i|a_i)$ takes the form in Fullerton and McAfee (1999).

Section 2 describes the model. Section 3 analyzes environments in which the contest is used exclusively as an incentive device. Section 4 considers more subtle objective functions, where the designer cares about more than just equilibrium actions. Section 5 examines design in the Fullerton and McAfee (1999) model in detail. Section 6 discusses implications and extensions, and Section 7 concludes.

2 Contests with stochastic performance

This section lays out the basic model and formulates the design problem.

2.1 Contest primitives

There is a fixed set $N = \{1, ..., n\}$ of risk neutral contestants or agents. Agent i takes costly action $a_i \in \mathbb{R}_+$. The cost function is $c_i(a_i)$, with $c(0) = 0$, $c_i'(a_i) > 0$, and $c_i''(a_i) \geq 0$. The action can typically be interpreted as effort and it influences the distribution of the agent's signal or performance, q_i. The distribution function is written $G_i(q_i|a_i)$. This is atomless whenever $a_i > 0$, in which case it has density $g_i(q_i|a_i) > 0$ and support $[\underline{q}_i, \overline{q}_i]$, which may or may not be bounded above or below. Note that the support is the same for all strictly positive actions. If $a_i = 0$, the possibility that the distribution is degenerate at $q_i = \underline{q}_i$ is allowed. Given actions, agents' signals are statistically independent.

The designer can award up to m identical and indivisible prizes, with $n > m \geq 1$. Agent i assigns value $v_i > 0$ to winning a prize. Each agent can win at most one prize and the value of losing is zero. There is no entry fee and the outside option is worth zero. Thus, the participation constraint is trivially satisfied because $a_i = 0$ guarantees non-negative payoff.

2.2 Assignment rules and the moral hazard problem

Let Ω denote the collection of all permissible sets of winners, and let ω denote an element of Ω. Thus, ω describes an assignment of prizes. Two cases are considered. In the first, the designer is forced to award all prizes and any $\omega \in \Omega$ must have precisely m (distinct) members. In the second case, the designer has complete freedom to decide how many of the m prizes to allocate. Thus, the only restriction is that $\omega \in \Omega$ has at most m members. These are the "no rationing" and "rationing" cases, respectively. Extensions are discussed in Section 6.

A contest elicits effort from agents. Hence, designing a contest is at heart a moral hazard problem. Thus, familiar logic can be applied. First, it is assumed that the entire performance profile is observed. A biased contest is then one in which the winners are not necessarily the agents with the highest performance.

Let $P_\omega(\mathbf{q})$ denote the probability that the group $\omega \in \Omega$ wins, given the performance profile $\mathbf{q} = (q_1, q_2, ..., q_n)$. Let $\mathbf{P} = \{P_\omega\}_{\omega\in\Omega}$ denote the ensuing "assignment rule." This is the endogenous design instrument. The designer can credibly and fully commit to any feasible assignment rule. The feasibility constraints are that $P_\omega(\mathbf{q}) \in [0, 1]$ for all $\omega \in \Omega$ and that $\sum_{\omega\in\Omega} P_\omega(\mathbf{q}) = 1$, for all $\mathbf{q}$.

While it is easiest to think of prizes as being identical and indivisible, the model does permit another interpretation. Let $m = 1$ denote the size of a perfectly divisible prize (e.g. the wage budget) and let $\Omega = N \cup \{0\}$. Then, $P_{\{i\}}(\mathbf{q})$ can be interpreted as the share of the prize that agent i receives, with $P_{\{0\}}(\mathbf{q})$ being the share that the designer retains for herself.

With some abuse of notation, the probability that agent i wins a prize, given $\mathbf{q}$, is

$$P_i(\mathbf{q}) = \sum\nolimits_{\{\omega\in\Omega|i\in\omega\}} P_\omega(\mathbf{q}). \tag{1}$$

Note that different assignment rules may yield the same reduced winning probability $P_i(\mathbf{q})$ when there are multiple prizes. Let $\mathbf{q}_{-i}$ and $\mathbf{a}_{-i}$ denote the performance profile and action profile of agent i's rivals, respectively. Given $\mathbf{a}_{-i}$, agent i's expected utility from action a_i is now

$$U_i(a_i, \mathbf{a}_{-i}|\mathbf{P}) = v_i \int \left(\int P_i(q_i, \mathbf{q}_{-i}) g_i(q_i|a_i) dq_i \right) \prod_{j\neq i} g_j(q_j|a_j) d\mathbf{q}_{-i} - c_i(a_i), \tag{2}$$

since signals are statistically independent. The factor after v_i integrates out the uncertainty over performance profiles and expresses the winning probability as a function only of the action profile. In the language of contest theory, this is the CSF. The CSF is endogenized by manipulating the assignment rule.

For an action profile $\mathbf{a} = (a_1, a_2, ..., a_n)$ to be implementable, it must constitute a Nash equilibrium of the contest game. Attention is restricted to pure strategy implementation throughout.

2.3 The contest environment

The analysis relies on the standard first-order approach, the validity of which places technical restrictions on $G_i(q_i|a_i)$. Thus, assume from now on that actions are continuous and that $g_i(q_i|a_i)$ is differentiable with respect to a_i when $a_i > 0$.

The likelihood-ratio,

$$L_i(q_i|a_i) = \frac{1}{g_i(q_i|a_i)} \frac{\partial g_i(q_i|a_i)}{\partial a_i},$$

plays an important role. Since repeated reference to the likelihood-ratio is necessary, the abbreviation LR will be used in the text. The LR is assumed to be strictly increasing in q_i. This is the *monotone likelihood-ratio property* (MLRP).

Assumption (MLRP): $L_i(q_i|a_i)$ is strictly increasing in q_i for all $a_i \in \mathbb{R}_{++}$ and all $i \in N$.

The MLRP implies that a higher action makes a lower performance less likely. In the standard contracting literature, the MLRP ensures that wage schedules are monotonic in signals. It plays a similar role here, guaranteeing that the optimal $P_i(q_i, \mathbf{q}_{-i})$ is non-decreasing in q_i. Rogerson (1985) adds a *convexity of the distribution function condition* (CDFC) that assumes that $G_i(q_i|a_i)$ is convex in a_i. The CDFC implies that the term in the parenthesis in (2) is concave in a_i for any monotonic $P_i(\mathbf{q})$. Thus, $U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})$ is concave in a_i.

Assumption (CDFC): $G_i(q_i|a_i)$ is convex in a_i for all $q_i \in [\underline{q}_i, \overline{q}_i]$ and all $i \in N$.

3 Assignment independent contests

This section begins the analysis by solving a tractable yet flexible class of contests. The next section considers a more general contest environment.

3.1 The objective function and the design problem

The designer is endowed with a Bernoulli utility function, $\pi(\mathbf{q}, \mathbf{a})$, which is allowed to depend on the performance profile and the action profile. The main restriction is that the designer does not care about the identity of the winners or the assignment itself. In other words, her preferences are *assignment independent*.

The designer may care directly about the action profile. The most common assumption in the contest literature is that she wishes to maximize total effort, or $\pi(\mathbf{q}, \mathbf{a}) = \sum_{j \in N} a_j$. For instance, a_i may capture human-capital accumulation that is important in the long run, whereas q_i is performance in the short run that is of lesser or no value but is readily observable.

However, the designer may also care directly about performance. For instance, consider a contest for one or more promotions among salesmen akin to Lazear and Rosen (1981). Here, the employer is presumably not directly interested in the salesmen's efforts but rather in the total volume of sales, or $\pi(\mathbf{q}, \mathbf{a}) = \sum_{j \in N} q_j$.

The commitment problem is less severe with assignment independent preferences, as the designer has no ex post incentive to deviate from the promised assignment rule. The designer aims to design the assignment rule $\mathbf{P}$ to implement an action profile $\mathbf{a}$ that maximizes her expected utility,

$$U_0(\mathbf{a}) = \mathbb{E}[\pi(\mathbf{q}, \mathbf{a})|\mathbf{a}].$$

The analysis proceeds under the assumption that $U_0(\mathbf{a})$ is monotonic.

Definition (AIM Contests): A contest is said to be Assignment Independent and Monotonic (AIM) if $U_0(\mathbf{a})$ is strictly increasing in a_i for all $i \in N$.

In any AIM contest, any optimal action profile is on the frontier of the set of implementable or feasible action profiles. Thus, the incentive compatibility problem takes centre stage. Therefore, these contests are the ideal starting point for understanding the incentive problem and how contest design incentivizes agents.

Two central messages emerge. First, for any frontier action there is an essentially unique incentive compatible assignment rule. That is, incentive compatibility more or less dictates contest design. However, the designer is left to decide which exact frontier action to induce. Second, the fundamental structure of the contest design is the same for all frontier actions. Hence, the principles underlying contest design is the same in all AIM contests.

3.2 Maximal individual and group effort

To understand incentives, it is useful to start by focusing on one given agent in isolation. Given (2), the marginal return to a small increase in a_i is

$$\frac{\partial U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i} = v_i \int \left(\int P_i(q_i, \mathbf{q}_{-i}) L_i(q_i|a_i) g_i(q_i|a_i) dq_i \right) \prod_{j \neq i} g_j(q_j|a_j) d\mathbf{q}_{-i} - c_i'(a_i). \tag{3}$$

Since the expected value of $L_i(q_i|a_i)$ is zero, the MLRP implies that $L_i(q_i|a_i)$ is negative for small q_i and positive for large q_i. It is clear that (3) is maximized if the prize is assigned to agent i if and only if $L_i(q_i|a_i)$ is positive. When $L_i(q_i|a_i)$ is positive, a marginal increase in a_i makes it more likely that a performance close to q_i is realized. There is no better carrot than promising the agent a prize for such performances and no better stick than to deny him a prize for performances that become less likely if his action increases. All proofs are in the Appendix.

Proposition 1 *Let $\overline{a}_i$ denote the highest action that agent i can be induced to take. If $\overline{a}_i > 0$ then there is an essentially unique $P_i(q_i, \mathbf{q}_{-i})$ function that induces $\overline{a}_i$. This takes the form of a threshold rule,*

$$P_i(q_i, \mathbf{q}_{-i}) = \begin{cases} 1 & \text{if } q_i \geq \widehat{q}_i(\overline{a}_i) \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

where $\widehat{q}_i(a_i)$ denotes the unique value of q_i for which $L_i(q_i|a_i) = 0$.[5]

Next, consider a contest with a group of $n_1 \leq n$ agents that are identical to agent 1. Assume that the designer must treat these agents symmetrically and must induce them to take the same action. In fact, it turns out that the optimal way to induce identical actions within the group is to use a symmetric design.

Let $\overline{a}_1^s \leq \overline{a}_1$ denote the highest implementable symmetric action in the group and assume that $\overline{a}_1^s > 0$. To induce this action, the same logic as before dictates that positive LRs should be rewarded if at all possible. However, other agents in the group may also have positive LRs, and there are only so many prizes to go around. Thus, arrange the n_1 agents in a line from the agent with the highest LR to the agent with the lowest. If rationing is allowed, then allocate a prize only to those of the first $\min\{n_1, m\}$ agents in line that have positive LRs. If rationing is not allowed, a threat can instead be made to allocate prizes to agents outside the group. There are $n - n_1$ other agents and m prizes, so the designer will still have to allocate at least $\max\{0, m - (n - n_1)\}$ prizes to the group. Thus, it may be necessary to allocate a prize to a group member with a negative LR,

[5]Here, $P_i(q_i, \mathbf{q}_{-i})$ is "essentially unique" because changes on a set of $\mathbf{q}$ of measure zero are irrelevant. The proof of the proposition outlines a method to characterize $\overline{a}_i$.

but members are still served in priority of their LR. The formal proof is omitted, but the argument is the same as in the proof of Theorem 1 in the next section.

This process generally only pins down part of the assignment rule since it does not specify what to do with prizes that are not allocated to the group in question. Thus, there is typically still some design freedom to motivate other agents.

3.3 Optimal design with two groups

Assume now that there are exactly two groups. There are n_1 contestants that are identical to agent 1 and n_2 that are identical to agent 2, $n_1 + n_2 = n$. The designer is restricted to inducing group-symmetric actions. The frontier of the feasible set is characterized and interpreted next.

A natural starting point is to identify the "corners" of the feasible set. The highest action that agents in group 1 can be induced to take is $\overline{a}_1^s$. Implementing this action ties the designer's hands to use the criteria described after Proposition 1. For agents in group 1, the performance of group 2 is irrelevant. Conversely, agents in group 2 fight over the scraps left by group 1. The two groups are essentially considered in sequence. This resembles the preference given to Canadian candidates on the academic job market, as described in the introduction. An action profile at the corner of the feasible set is optimal if the objective is to maximize the highest action, or $\pi(\mathbf{q}, \mathbf{a}) = \max\{a_1, a_2, ..., a_n\}$. Note, however, that this objective function is not strictly monotonic.

Let $\underline{a}_2^s$ denote the highest action that agents in group 2 can be induced to take, given agents in group 1 are induced to take action $\overline{a}_1^s$. If rationing is ruled out, any prizes that remain after group 1 has been served must be awarded to the agents in group 2 with the highest LRs. If rationing is permitted, then agents with negative LRs are excluded.

To illustrate, consider the case with a single prize. Then, without rationing, $\underline{a}_2^s$ is smallest when $n_2 = 1$. This is because the only member of group 2 wins if and only if all other contestants have negative LRs. This is an event that the agent cannot influence, and thus $\underline{a}_2^s = 0$ is the best response. With more members of group 2, within-group competition can incentivize $\underline{a}_2^s > 0$. In contrast, when rationing is possible, $\underline{a}_2^s$ is largest when $n_2 = 1$. If the prize is not claimed by an agent in group 1, the only agent in group 2 wins if and only if his LR is positive. This maximizes his incentives. When $n_2 \geq 2$, competition from other agents muddle incentives by lowering the probability that he wins if his LR is positive.

Finally, consider the parts of the frontier that are not at the corners. Here, $a_1 \in (\underline{a}_1^s, \overline{a}_1^s)$ and $a_2 \in (\underline{a}_2^s, \overline{a}_2^s)$, where $\underline{a}_1^s$ and $\overline{a}_2^s$ are defined analogously to $\underline{a}_2^s$ and $\overline{a}_1^s$, respectively. The design must now balance incentives, and therefore compare LRs, across groups. To do so, give agent i with performance q_i an endogenous "score" of

$$s_i(q_i) = \mu_i v_i L_i(q_i|a_i),$$

and prioritize agents in the order of their scores.

Theorem 1 *Consider a contest with two groups, and assume that group-symmetric actions must be implemented. Without rationing, any action profile* $\mathbf{a}$ *that is on the frontier of the feasible set with* $a_1 \in (\underline{a}_1^s, \overline{a}_1^s)$ *and* $a_2 \in (\underline{a}_2^s, \overline{a}_2^s)$ *is implemented by assigning prizes to the* m *agents with the highest scores. With rationing, prizes are assigned to the agents with the highest positive scores, up to at most* m *prizes. In either case,* $\mu_i \in (0, \infty)$, $i \in N$, *is endogenously determined and group-symmetric, and the resulting assignment rule is essentially unique.*

As $\frac{\mu_1}{\mu_2} \to \infty$, an agent in group 1 with a positive (negative) LR is more or less guaranteed to outscore (be outscored by) any agent in group 2. Hence, the design approaches the design that implements $\overline{a}_1^s$. More generally, $\frac{\mu_i}{\mu_j}$ is calibrated to ensure compliance, or incentive compatibility. Hence, $\frac{\mu_i}{\mu_j}$ is best thought of as a measure of how strongly the designer pushes agents in group i relative to agents in group j to work hard. It is thus not necessarily a measure of favoritism. Which group of agents is favored with higher winning probabilities is more subtle and is discussed in Section 3.3.3.

Theorem 1 implies that irrespective of the exact form of AIM preferences, the structure of the optimal contest is remarkably robust. The remainder of this section first highlights and illustrates two specific properties of the optimal design. It then discusses how the designer's objective function impacts the optimal action profile and the optimal design.

3.3.1 Rationing and minimum standards for eligibility

When rationing is allowed, agents with negative scores or LRs are disqualified. In other words, $\widehat{q}_i(a_i)$ as defined in Proposition 1 can be seen as a *minimum standard for eligibility*. At a selective college, this is the minimum admission standard. In a promotion contest, it is the bar that the agent must clear to be considered for a promotion. In an innovation contest, it is the standard below which the innovation is deemed unqualified for consideration, e.g. a drug that fails clinical trials. Note that prizes are withheld with positive probability in equilibrium. Since rationing impacts the design, it is clearly of value to the designer.

The distributions of performance impact the size and stringency of the minimum standards and how they depend on actions and compare across groups. To illustrate, the consequences of a mild regularity condition due to Chade and Swinkels (2020) are considered. Given that $\frac{\partial G_i(q_i|a_i)}{\partial a_i}$ is negative as an implication of the MLRP, their condition, the no-upward-crossing condition (NUC), can be expressed as the requirement that $-\frac{\partial G_i(q_i|a_i)}{\partial a_i}$ is log-supermodular in a_i and q_i.

Proposition 2 *If $G_i(q_i|a_i)$ satisfies NUC then, along the frontier of the feasible set, the minimum standard is weakly increasing in the action that agent i is induced to take. That is, $\widehat{q}_i(a_i)$ is weakly increasing in a_i.*

Thus, if the two groups of agents draw their performance from the same family of distributions, or $G_i(q_i|a_i) = G(q_i|a_i)$, $i = 1, 2$, then the group that faces the most stringent standard supplies higher effort in equilibrium. However, this need not be true when distributions are different across groups. Note also that since actions move in opposite directions along the frontier of the feasible set, minimum standards must likewise move in opposite directions when NUC is satisfied.

Example 1 (Minimum standards for eligibility in two models): Two models drawn from the classic principal-agent literature are considered. First, Grossman and Hart (1983) introduce a *spanning condition*, such that

$$G_i(q_i|a_i) = f_i(a_i)H_i(q_i) + (1 - f_i(a_i))K_i(q_i),\ q_i \in [\underline{q}_i, \overline{q}_i], \tag{5}$$

where H_i and K_i are distribution functions with densities h_i and k_i and where $f_i(a_i) \in [0, 1)$. The MLRP and CDFC are satisfied if $f_i'(a) > 0 \geq f_i''(a_i)$ and $\frac{h_i(q_i)}{k_i(q_i)}$

is strictly increasing. This implies that H_i first-order stochastically dominates K_i. Think of $f_i(a_i)$ as being the amount of time or resources that the agent devotes to using the more productive technology H_i. Here, NUC is satisfied because $\ln\left(-\frac{\partial G_i(q_i|a_i)}{\partial a_i}\right)$ is additively separable in q_i and a_i. In fact, $\widehat{q}_i(a_i)$ is located where $\frac{h_i(q_i)}{k_i(q_i)} = 1$. That is, the minimum standard $\widehat{q}_i(a_i)$ is *independent* of a_i. This has several important implications. First, the minimum standard is the *same* in *all* AIM contests. Second, the minimum standard is independent of v_i, implying that if technologies are symmetric across groups then minimum standards are symmetric across groups even if valuations differ. Finally, the agent is more likely to pass the standard the higher his equilibrium action is.

Second, consider the *best-shot model*, in which

$$G_i(q_i|a_i) = H_i(q_i)^{f_i(a_i)},\ q_i \in [\underline{q}_i, \overline{q}_i], \tag{6}$$

where $H_i(q_i)$ is a distribution function. Thus, $G_i(q_i|a_i)$ is the distribution of the best draw from $H_i(q_i)$ – the best-shot – out of a total of $f_i(a_i) \geq 0$ draws. In an innovation contest, H_i can be interpreted as the distribution of the quality of a single idea and f_i as the number of ideas. The MLRP, CDFC, and NUC all hold if $f_i'(a_i) > 0 \geq f_i''(a_i)$. Rogerson (1985) mentions a special case and Fullerton and McAfee (1999) likewise rely on a special case. Here, $\widehat{q}_i(a_i)$ is determined where $G_i(\widehat{q}_i(a_i)|a_i) = e^{-1}$. Thus, the minimum standard is increasing in a_i, but the *probability* of passing the standard is always the same and equals $1 - e^{-1}$. It follows that how many prizes are allocated has the *same* distribution in *all* AIM contests, *regardless* of valuations and group differences in H_i and f_i. For instance, the probability that zero prizes is awarded in e^{-n}. $\square$

An unbiased contest with an identity-independent minimum standard is optimal if all agents are symmetric and their actions enter the designer's utility function symmetrically. Such a contest induces higher actions than an unbiased contest with no minimum standard. It follows that if the contest is perturbed slightly, such that v_1 and v_2 are allowed to differ slightly, then an identity-independent minimum-standard remains an improvement over an unbiased contest with no rationing. Thus, rationing is of value even if the designer is unable to completely fine-tune the design by dictating identity-dependent scoring functions.

3.3.2 Thresholds of excellence and first claimants

Recall that if $a_1 = \overline{a}_1^s$ is induced, then any agent in group 1 is guaranteed to outscore all agents in group 2 as long as his LR is positive. In other words, an agent in group 1 whose performance is sufficiently high does not have to worry about competition from outside his group.

Consider next action profiles away from the corners, as in Theorem 1, but assume that LRs are bounded above. This assumption holds in the best-shot model, and typically also in any model where performance is bounded above. Then, the maximum scores $s_1(\overline{q}_1)$ and $s_2(\overline{q}_2)$ in Theorem 1 are finite. More to the point, the maximum scores are also generically distinct since it would be entirely coincidental if μ_1 and μ_2 happen to equate $s_1(\overline{q}_1)$ and $s_2(\overline{q}_2)$. To fix ideas, imagine that $s_1(\overline{q}_1) > s_2(\overline{q}_2)$. Then, an agent in group 1 with sufficiently high performance is guaranteed to outscore all agents in group 2. In short, the design once again features a *threshold of excellence* above which members of group 1 cannot lose to a member of group 2. An agent in group 1 who passes the threshold of excellence is guaranteed a prize if $n_1 \leq m$. If $n_1 > m$, the agent may lose but only if he is outscored by sufficiently many agents in his own group. Thus, the design is close to the kind of guaranteed admission that is sometimes given to the best in-state students. It is as if agents in group 1 who passed the threshold are "first claimants" of prizes. Other agents receive a prize only if there are prizes left over once the first claimants are served.

Which group is promised the status of first claimants depends on which action profile the designer is seeking to implement. If an action profile near $(\overline{a}_1^s, \underline{a}_2^s)$ is implemented, then $\frac{\mu_1}{\mu_2}$ is very large and $s_1(\overline{q}_1) > s_2(\overline{q}_2)$. Intuitively, the prospect of passing a threshold of excellence and being insulated from competition from the other group is an effective incentive device. Formally, the reason is that this is the best way of rewarding large LRs. However, the optimal action profile depends on the designer's objective function, as discussed next.

3.3.3 Objective functions and favoritism

This subsection illustrates how the optimal action profile depends on the designer's objective function by considering two extreme examples. These examples also demonstrate that which group of agents is "favored" depends on the distributions of performance and in particular on the magnitude of $\widehat{q}_i(a_i)$.

It has already been observed that if the objective is to maximize the highest action, or $\pi(\mathbf{q}, \mathbf{a}) = \max\{a_1, a_2, ..., a_n\}$, then it is optimal to induce an action profile at the corner of the feasible set. When actions $\overline{a}_1^s$ and $\underline{a}_2^s$ are induced, members of group 1 are first claimants. Even though this may appear to place group 1 in an advantageous position, its members in fact need not win more often or be better off than agents in group 2.

Example 2 (Incentives versus outcomes): There is a single prize and two agents with the same valuation. Rationing is ruled out. When $(a_1, a_2) = (\overline{a}_1^s, \underline{a}_2^s) = (\overline{a}_1, 0)$ is induced, agents 1 and 2 win with probability $1 - G_1(\widehat{q}_1(\overline{a}_1)|\overline{a}_1)$ and $G_1(\widehat{q}_1(\overline{a}_1)|\overline{a}_1)$, respectively. Thus, agent 2 wins most often if $G_1(\widehat{q}_1(\overline{a}_1)|\overline{a}_1) > \frac{1}{2}$. This necessitates that $\widehat{q}_1$ is above the median performance, but nothing precludes this from happening. In this case, agent 2 is also better off, since he incurs lower effort costs. Agents who exert higher effort need not have higher winning probabilities because incentives derive from the action's *marginal* impact on the expected winning probability, not on the *level* of the winning probability itself. The optimal design manipulates the former with no regard to the latter.

As an example, assume that $v_i = 1$ and $c_i(a_i) = a_i$, $i = 1, 2$. Then, only actions below 1 can be rationalized. Thus, for $a_i \in [0, 1)$, assume that

$$G_i(q_i|a_i) = \sqrt{a_i} q_i^4 + (1 - \sqrt{a_i})\, q_i, \;\; q_i \in [0, 1]\,. \tag{7}$$

This is a special case of (5). The MLRP and the CDFC are satisfied. Here, $\overline{a}_1 = 0.056$ and $\widehat{q}_i(\overline{a}_1) = 0.63$, while the median performance is 0.61. Agent 1 wins with probability 0.482. $\square$

On the other hand, a more balanced action profile is optimal when the designer considers the agents' actions to be complementary. In the extreme case where actions are perfect complements, or $\pi(\mathbf{q}, \mathbf{a}) = \min\{a_1, a_2, ..., a_n\}$, it is optimal to induce symmetric actions, or $a_1 = a_2$, whenever possible. The next result describes some of the key characteristics of the optimal design in this case.

Proposition 3 *Assume that $v_1 > v_2$ and that technologies are symmetric across* *groups, or $G_i(q_i|a_i) = G(q_i|a_i)$ and $c_i(a_i) = c(a_i)$, $i = 1, 2$. If an action profile with $a_2 = a_1$ is on the frontier of the feasible set, then it is implemented with $\mu_2 v_2 > \mu_1 v_1$. If rationing is allowed, then any member of group 2 wins with a higher probability than any member of group 1.*

Agents with lower valuation need a stronger push to work as hard as agents with higher valuations, which is why $\mu_2 v_2 > \mu_1 v_1$ in Proposition 3. This increases the score of an agent in group 2 if his LR is positive, but decreases it if his LR is negative. Thus, without rationing, these confounding effects mean that he may or may not win more often. However, only the first effect is relevant when rationing is possible. Then, winning probabilities are unambiguously higher for agents in group 2. Thus, contests with rationing give "cleaner" predictions.

To prove that either group may win more often without rationing, consider a two-agent/one-prize contest with symmetric technologies. A conceptual trick from Jung and Kim (2015) is employed. They suggest transforming the problem in such a way that the distribution of the LR is thought of as the primitive of the problem. Thus, let $G_L(L_i|a_i)$ denote the identity-independent distribution of the LR and $g_L(L_i|a_i)$ its density, $i = 1, 2$.

Corollary 1 *Assume that $n_1 = n_2 = m = 1$, technologies are symmetric across groups, and that rationing is ruled out. If $v_1 > v_2$ and an action profile with $a_2 = a_1 = a^*$ on the frontier of the feasible set is implemented, then agent 2 wins more (less) often than agent 1 if $G_L(L|a^*)$ is convex (concave) in L.*

It can be verified that $G_L(\cdot|a^*)$ is convex in the best-shot model. However, it is possible to construct examples where it is concave in the spanning model. Thus, which agent wins more often is sensitive to the model. With reference to Example 2, recall that $L_i(\widehat{q}_i(a^*)|a^*) = 0$ is the mean of $G_L(\cdot|a^*)$, and that the median is above (below) the mean when $G_L(\cdot|a^*)$ is convex (concave). For instance, $\widehat{q}_i(a_i)$ is below the median in the best-shot model, where $G_i(\widehat{q}_i(a_i)|a_i) = e^{-1} \approx 0.368$.

Proposition 3 implies that $s_2(\overline{q}_2) > s_1(\overline{q}_1)$ when technologies are symmetric and $a_1 = a_2$. Thus, the "weaker" agents in group 2 are made first claimants. Similarly, when rationing is permitted, the minimum standard is the same across groups. The reason is that $\widehat{q}_1(a_1) = \widehat{q}_2(a_2)$ since $a_1 = a_2$ and technologies are symmetric. However, since agents in the weaker group receive more generous scores whenever they pass the minimum standard, they have higher winning probabilities and therefore tend to be over-represented among the winners.

However, heterogenous minimum standards can be explained if technologies are asymmetric, even when $a_1 = a_2$. For instance, consider the best-shot model but assume that group 2 is less productive than group 1, with $H_1(q) \leq H_2(q)$ for all q and/or $f_1(a) \geq f_2(a)$ for all a. Then, group 2 will face a lower minimum standard than group 1 if the two are induced to take similar actions. This is consistent with lowering admission standards for disadvantaged applicants.

The action profile in Proposition 3 is close to optimal when actions are close to perfect complements. On the other hand, agents with higher valuations are easier to incentivize, and so it is intuitive that if actions are closer to perfect substitutes then the optimal design will be geared more towards incentivizing the strong group of agents. Section 5 considers this in detail in the best-shot model.

4 Costly and separable contests

The assignment rule is dictated by the agents' incentive compatibility constraints when an action profile on the frontier is implemented. This lack of flexibility may prove costly to the designer if her preferences are not assignment independent. In principle, it may be better to induce an action profile that is not on the frontier of the feasible set. These actions can be induced in many ways, meaning that the assignment rule can now better reflect the designer's objectives.

Thus, the designer's Bernoulli utility is now allowed to depend on the assignment and it is therefore written $\pi_\omega(\mathbf{q}, \mathbf{a})$ in the event that the assignment is ω. This is assumed to be separable in the sense that it takes the form

$$\pi_\omega(\mathbf{q}, \mathbf{a}) = \pi(\mathbf{q}, \mathbf{a}) + \upsilon_\omega(\mathbf{a}), \quad \omega \in \Omega. \tag{8}$$

The new term $\upsilon_\omega(\mathbf{a})$ incorporates the designer's preferences over the assignment. For example, she may care more about the actions of the winners than the losers, e.g. $\upsilon_\omega(\mathbf{a}) = \sum_{i \in \omega} a_i$. Similarly, $\upsilon_\omega(\mathbf{a})$ can capture the cost to the designer of awarding prizes, with $\upsilon_\omega(\mathbf{a}) = -C(|\omega|)$, where C is a cost function.

Preferences are assumed to be monotonic in actions, or

$$\frac{\partial\left(U_0(\mathbf{a})+\upsilon_\omega(\mathbf{a})\right)}{\partial a_i} \geq 0,\ \forall i \in N, \text{ with } \frac{\partial\left(U_0(\mathbf{a})+\upsilon_\omega(\mathbf{a})\right)}{\partial a_i} > 0,\ \forall i \in \omega, \tag{9}$$

where $U_0(\mathbf{a}) = \mathbb{E}[\pi(\mathbf{q},\mathbf{a})|\mathbf{a}]$ as before. Thus, for any assignment, the designer is better off if agents take higher actions.

Definition: A contest is Separable and Monotonic if (8) and (9) hold.

The optimal design is again characterized for contests with two groups of agents. For brevity, focus is also on contests where any second-best action profile is interior, or $a_i > 0$ for all $i \in N$, but not on the frontier of the feasible set. That is not to say that the second-best action cannot be on the frontier of the feasible set, but in that case the design must take the form in Theorem 1.

Ideally, the designer would prefer to select an assignment that maximizes $\upsilon_\omega(\mathbf{a})$. Thus, each possible assignment, ω, is assigned an aggregate score,

$$s_\omega(\mathbf{q}) = \upsilon_\omega(\mathbf{a}) + \sum\nolimits_{i\in\omega} \mu_i v_i L_i(q_i|a_i). \tag{10}$$

The score is a compromise between two considerations. First, $\upsilon_\omega(\mathbf{a})$ is relevant to the designer but it ignores incentive compatibility. The second term is present to "fix" this problem and ensure incentive compatibility.[6,7] The assignment with the highest score is chosen, and its members receive prizes. Note that the empty assignment – when no prizes are awarded – is assigned a score of $\upsilon_\emptyset(\mathbf{a})$. Thus, when rationing is allowed, all prizes are withheld if all other assignments yield scores below $\upsilon_\emptyset(\mathbf{a})$. AIM contests are special cases, with $\upsilon_\omega(\mathbf{a}) = 0$ for all ω.

Proposition 4 *Consider a Separable and Monotonic contest with two groups, and assume that group-symmetric actions must be implemented. Assume also that $\upsilon_\omega(\mathbf{a})$ is group-symmetric.*[8] *Any second-best action profile* $\mathbf{a}$ *that is interior*

[6] Action profiles near the frontier presumably have large μ_i's. As the μ_i's goes to infinity, the $\upsilon_\omega(\mathbf{a})$ terms lose their significance and the assignment rule converges to that in Theorem 1. Note that the μ_i's can explode while their ratios converge to that implied by Theorem 1.

[7] In a more general model where υ_ω depends on $\mathbf{q}$, the agents' first-order conditions produce a similar scoring rule to $s_\omega(\mathbf{q})$, but it is harder to verify that second-order conditions are satisfied.

[8] Thus, removing an agent from ω and replacing him with another agent from the same group and with the same action does not change $\upsilon_\omega(\mathbf{a})$.

and not on the frontier of the feasible set is optimally implemented by choosing the assignment in Ω with the highest aggregate score, where $\mu_i \in (0,\infty)$, $i \in N$, is endogenously determined and group-symmetric.

Consider a contest with costly prizes and $v_\omega(\mathbf{a}) = -C(|\omega|)$. Here, the aggregate score of assignment ω is $s_\omega(\mathbf{q}) = \sum_{i\in\omega} \mu_i v_i L_i(q_i|a_i) - C(|\omega|)$. The highest scoring assignment can then be found by comparing "marginal revenue" and "marginal costs". First, arrange agents in descending order according to their individual scores, $\mu_i v_i L_i(q_i|a_i)$, which now play the role of marginal revenue. Then, prizes are awarded as long as marginal revenue exceeds the corresponding marginal cost. Thus, if x prizes have already been awarded, an additional prize should be handed out if the $(x+1)st$ highest individual score exceeds $C(x+1) - C(x)$, where the latter captures marginal costs. Appendix B explores how the optimal action profile depends on costs by focusing on contests with symmetric agents.

The specification in (8) allows the designer to have preferences over the identity of the winners. A government may desire that the distribution of students admitted into university reflects the distribution of various ethnicities in the population. With two sets of agents, N_1 and N_2, the utility function may then take the form

$$\pi_\omega(\mathbf{q},\mathbf{a}) = d\left(|N_1 \cap \omega|, |N_2 \cap \omega|\right) + \beta \textstyle\sum_{i\in N} a_i,$$

where $d(\cdot,\cdot)$ is a function of how many agents from each group is admitted, and where $\beta > 0$ reflects how important actions are to the designer. Here, the aggregate score of any given assignment depends on its composition. Thus, the winners need not be the agents with the highest individual scores, $\mu_i v_i L_i(q_i|a_i)$.

For a final example, assume that $c_i(a_i) = a_i$, $i \in N$, and that the designer cares about both social welfare and total effort, with

$$\begin{aligned}\pi_\omega(\mathbf{q},\mathbf{a}) &= \gamma\left(\textstyle\sum_{i\in\omega} v_i - \sum_{i\in N} a_i\right) + (1-\gamma)\sum_{i\in N} a_i \\ &= \gamma \textstyle\sum_{i\in\omega} v_i + (1-2\gamma)\sum_{i\in N} a_j\end{aligned}$$

where $\gamma \in [0, \frac{1}{2})$ is the weight given to social welfare. Here, prizes are awarded to the agents with the highest amended scores, $\gamma v_i + \mu_i v_i L_i(q_i|a_i)$. Epstein, Mealem, and Nitzan (2011) utilize the approach in Section 5.3 to analyze a similar contest.

5 The best-shot model with a single prize

The analysis has identified general properties of optimal contest design. These design principles are general in the sense that they are the same for all distribution functions and for a wide variety of objective functions. This section specializes the setting to a single-prize contest with best-shot performance technologies, as in (6). This makes it possible to illustrate the approach and to discuss the role of the objective function in more detail (Sections 5.1 and 5.2). The best-shot model is also ideally suited to contrast and compare the paper's approach to a popular but less solidly founded approach to contest design (Section 5.3).

Given (6), the "impact function" $f_i(a_i)$ and the cost function $c_i(a_i)$ are instrumental in determining incentives. Thus, define

$$t_i(a_i) = c_i'(a_i)\frac{f_i(a_i)}{f_i'(a_i)}$$

as a measure of marginal costs relative to marginal productivity. Intuitively, when this is small the agent has a greater incentive to increase his action. It turns out that t_i captures everything that is important about the agent's technology. By concavity of f_i and convexity of c_i, t_i is a strictly increasing function.

For simplicity, assume that $\lim_{a_i \to 0} t_i(a_i) = 0$. Roughly speaking, this means that $a_i = 0$ is optimal to the agent only when his action cannot influence the outcome. The condition is satisfied if $f_i(0) = 0$ and/or $c_i'(0) = 0$.

5.1 Properties of the feasible set

As before, assume that there are two groups and that the designer must use group-symmetric rules. To start, rationing is ruled out. Assume that the contest is an AIM contest. Thus, the frontier of the feasible set is of interest. Given an equilibrium action profile $\mathbf{a}^*$, agent i's score in the best-shot model simplifies to

$$s_i(q_i) = \tau_i \left(1 + \ln G_i(q_i|a_i^*)\right), \tag{11}$$

where

$$\tau_i = \mu_i v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)}.$$

The score of a given performance relies crucially on where it sits in the equilibrium distribution, $G_i(\cdot|a_i^*)$. If $\tau_1 = \tau_2$, then the agent whose performance is at the highest quantile also obtains the highest score. Thus, in this case, any agent regardless of group membership is ex ante equally likely to win. If $\tau_i > \tau_j$, then $s_i(\overline{q}_i) > s_j(\overline{q}_j)$ and agents in group i are first claimants.

With two groups, only the relative sizes of τ_1 and τ_2 matter. Thus, for any $\frac{\tau_i}{\tau_j} \in (0, \infty)$, it is possible to derive the action profile that is being implemented.

Proposition 5 *Consider the best-shot model with two groups, group-symmetric rules, and no rationing. For any action profile that is on the frontier of the feasible set but not at a corner, the equilibrium action of an agent in group i, a_i^*, is determined by $\kappa_i = \frac{\tau_i}{\tau_j} \in (0, \infty)$, with*

$$t_i(a_i^*) = v_i F\left(\kappa_i | n_i, n_j\right), \tag{12}$$

and where

$$F\left(\kappa_i | n_i, n_j\right) = \begin{cases} e^{n_j(\kappa_i - 1)} \frac{n_j \kappa_i + n_i - 1}{\left(n_j \kappa_i + n_i\right)^2} & \text{if } \kappa_i \in (0,1) \\ \frac{n_i - 1}{n_i^2} + e^{n_i\left(\frac{1}{\kappa_i} - 1\right)} \frac{n_j}{n_i^2} \frac{n_j \kappa_i^2 + \kappa_i n_i (2 - n_j) - n_i^2}{\left(n_j \kappa_i + n_i\right)^2} & \text{if } \kappa_i \geq 1 \end{cases}.$$

Here, $F\left(\kappa_i | n_i, n_j\right)$ is strictly increasing in κ_i and satisfies $F(1|n_i, n_j) = \frac{n-1}{n^2}$. Hence, a_i^* is strictly increasing in κ_i.

The corner actions, $\underline{a}_i^s$ and $\overline{a}_i^s$, can be obtained by letting κ_i converge to zero or infinity, respectively. Away from the corners it holds that $\kappa_2 = \frac{1}{\kappa_1}$. Hence, for any $\kappa_1 \in (0, \infty)$, the equilibrium actions are determined by

$$t_1(a_1^*) = v_1 F\left(\kappa_1 | n_1, n_2\right) \text{ and } t_2(a_2^*) = v_2 F(1/\kappa_1 | n_2, n_1).$$

The frontier of the feasible set can now be traced out by varying the parameter κ_1. By Proposition 5, a_1^* is increasing in κ_1 while a_2^* is decreasing in κ_1. Note that the two groups generally speaking do not exert the same effort when $\kappa_1 = 1$.

Example 3 (Linear Technologies): Assume that $c_i(a_i) = f_i(a_i) = a_i$, $i = 1, 2$, in which case $t_i(a_i) = a_i$. Assume also that $v_1 = 2$, $v_2 = 1$. Figure 1

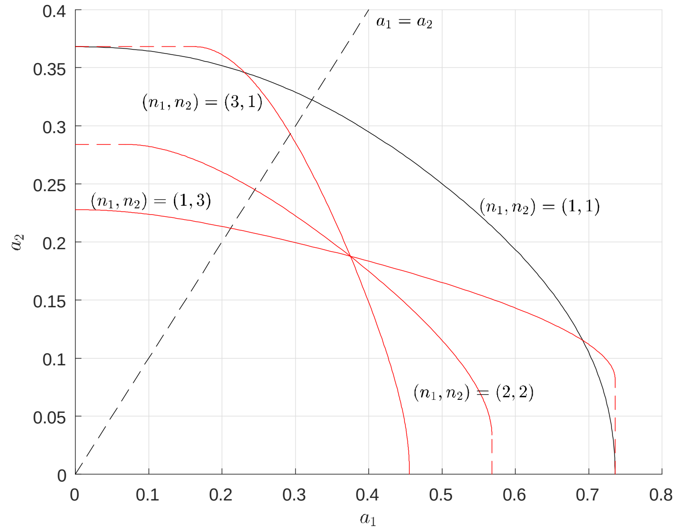

Figure 1: The frontier of the feasible set for different (n_1, n_2) with $n = 2$ (black curve) and $n = 4$ (red curves), given $v_1 = 2$ and $v_2 = 1$.

illustrates the feasible set and how it varies with n_1 and n_2. For instance, since $F(1|n_i, n_j)$ depends only on n, the frontier pivots when n_1 and n_2 change in opposite directions to maintain the same n. $\square$

The feasible set in Figure 1 is convex. More generally, this holds whenever t_1 and t_2 are convex. For example, t_i is convex if, as commonly assumed, $c_i(a_i) = a_i^{\gamma_i}$, $\gamma_i \geq 1$, and $f_i(a_i) = \beta_i a_i^{r_i}$, where $\beta_i > 0$, $r_i \in (0, 1]$.

Lemma 1 *Letting $\kappa_1 = \frac{\tau_1}{\tau_2} \in (0, \infty)$, the marginal rate of transformation – i.e. the slope of the frontier of the feasible set – is given by*

$$\frac{da_2^*}{da_1^*} = -\frac{t_1'(a_1^*)}{t_2'(a_2^*)}\frac{v_2}{v_1}\frac{n_1}{n_2}\kappa_1,$$

The feasible set is a convex set if $t_1(a_1)$ and $t_2(a_2)$ are convex functions.

As explained after (11), any agent is equally likely to win if $\tau_1 = \tau_2$, or $\kappa_1 = 1$. When τ_1 increases, the scores of agents in group 1 pivots around zero, increasing when performance is at a high quantile and decreasing otherwise. The latter is less relevant because negative scores are not only less likely, occurring only with probability e^{-1}, but are also less likely to win in the first place. Hence, increases in τ_1 relative to τ_2 increases the overall probability that a member of group 1 wins. Recall that if $\tau_1 > \tau_2$, or $\kappa_1 > 1$, then $s_1(\overline{q}_1) > s_2(\overline{q}_2)$. Thus, agents in the group that benefits from a threshold of excellence are more likely to win ex ante.

Proposition 6 *Consider the best-shot model with two groups, group-symmetric rules, and no rationing. Then, along the frontier of the feasible set, the probability that a given agent in group i wins is strictly increasing in κ_i and equals $\frac{1}{n}$ if $\kappa_i = 1$. If $\kappa_i \geq 1$, a further increase in κ_i causes members of group i to pass the threshold of excellence more often in equilibrium.*

Expected utility of agents in group i increases in κ_i when $n_i \geq 2$. The higher winning probability more than compensates for the increased effort cost.[9] Thus, it is natural to think of κ_i as a measure of how "nice" the designer is to group i.

Corollary 2 *Members of group i are better off the higher κ_i is, whenever $n_i \geq 2$.*

Proposition 6 is silent on who wins ex post for a given performance profile. This is determined not only by τ_1 and τ_2, but also by the mappings from performances to quantiles. These are different across groups and are functions of equilibrium actions. Unraveling these relationships leads to the following comparison of scoring functions. The scoring functions capture the overt discrimination between groups that is visible to an outsider, but note that they may cross in such a way that no group is given an advantage across all performance levels.

Corollary 3 *Assume that $v_1 > v_2$ and that technologies are symmetric across groups, or $H_i(q_i) = H(q_i)$, $f_i(a_i) = f(a_i)$, and $c_i(a_i) = c(a_i)$, $i = 1, 2$. If rationing is ruled out, then there exists a $\kappa'_1 \in (0, 1)$ such that:*

- *If $\kappa_1 \in [\kappa'_1, 1]$ then $s_1(q) - s_2(q)$ is always weakly negative: Members of group 1 score weakly lower than members of group 2 with the same performance.*
- *If $\kappa_1 \in (0, \kappa'_1) \cup (1, \infty)$ then $s_1(q) - s_2(q)$ changes sign once: The group that are first claimants are advantaged if performances are high but disadvantaged if performances are low.*

[9] In Example 3, expected utility is u-shaped and minimized at $\kappa_i = \frac{1}{n_j} \leq 1$ if $n_i = 1$. Again, $n_i = 1$ is a special case because here agent i does not face any within-group competition.

5.2 Optimal action profiles and contest design

This subsection examines how the optimal action profile is determined under the simplifying assumption that technologies are symmetric across groups, implying that $t_i(a_i) = t(a_i)$, $i = 1, 2$. This makes it possible to focus on the roles of the (common) technology and of the designer's objective function.

Conceptually, the exercise is simple. First, identify the feasible set. Then, maximize the designer's utility, $U_0(\mathbf{a})$, over the feasible set. If the feasible set is convex and $U_0(\mathbf{a})$ is quasi-concave, then the solution is unique and given by the familiar tangency condition.[10] Such regular environments are considered first.

To illustrate, assume that the designer's objective is to maximize total effort. Invoking group symmetry, the utility function is then $n_1a_1+n_2a_2$ and the marginal rate of substitution (MRS) is $-\frac{n_1}{n_2}$. As in Example 3, assume also that $t(a_i) = a_i$. Then, by Lemma 1, the marginal rate of transformation (MRT) is $-\frac{v_2}{v_1}\frac{n_1}{n_2}\kappa_1$. Tangency occurs where

$$\kappa_1 = \frac{v_1}{v_2}. \tag{13}$$

Assuming $v_1 > v_2$, the optimal design satisfies $\kappa_1 > 1$. Thus, "stronger" agents are induced to work harder, but they also benefit from being first claimants and they win more often.

Moving towards generalizations, three cut-off values of κ_1 are important. The first is $\kappa_1 = \frac{v_1}{v_2} > 1$, as in (13). The second is $\kappa_1 = 1$, above which members of group 1 are first claimants and are more likely to win. The third threshold, $\kappa_1^s < 1$, is defined as the level of κ_1 that induces symmetric actions, or $a_1^* = a_2^*$. If the asymmetry is very large – with v_1 at least four times larger than v_2 – then there are (n_1, n_2) combinations for which $a_1^* > a_2^*$ for all κ_1. In this case, define $\kappa_1^s = 0$. In any case, κ_1^s is below κ_1' in Corollary 3. The interval $(\kappa_1^s, \frac{v_1}{v_2})$ thus contains all the possible configurations of scoring functions in Corollary 3.

In general, the comparison of the MRT and the MRS depends on the technology, valuations, and group sizes on the one hand and on the designer's objective function on the other. To start, maintain the assumption that the designer seeks

[10] The slope of the frontier at the corners is 0 or $-\infty$, respectively. This is illustrated in Figure 1 and follows more formally from Lemma 1. Thus, the solution is interior whenever $U_0(\mathbf{a})$ is monotonic and quasi-concave.

to maximize total effort but allow the technology to take more general forms.

Proposition 7 *Assume the designer's objective is to maximize total effort. Assume that $v_1 > v_2$ and $t_i(a_i) = t(a_i)$, $i = 1, 2$, with $t(a_i)$ convex. Let κ_1^{Σ} denote the unique optimal value of κ_1. Then, $\kappa_1^{\Sigma} \in (\kappa_1^s, \frac{v_1}{v_2}]$ and $a_1^* > a_2^*$ in equilibrium. If $t(a_i)$ is also log-concave, then $\kappa_1^{\Sigma} \geq 1$. However, if $t(a_i)$ is locally log-convex, then there are (v_1, v_2) valuations with $v_1 > v_2$ for which $\kappa_1^{\Sigma} < 1$, in which case any member of group 2 wins more often than any member of group 1.*

In Example 3 and in the example prior to Lemma 1, $t(a_i)$ is both convex and log-concave. Log-concavity intuitively rules out that $t(a_i)$ is "too convex". In comparison, consider a contest where $f(a) = a$, $c'(a) = e^{\frac{1}{2}a^2}$, in which case $t(a) = ae^{\frac{1}{2}a^2}$ is globally convex. However, it is log-concave for $a < 1$ and log-convex for $a > 1$. If $v_1 > v_2 > e^{-\frac{1}{2}}\frac{n^2}{n-1}$, then the optimal value of κ_1 is strictly smaller than 1. Here, marginal costs are "extremely convex," and it is therefore hard to entice even agents in group 1 to work hard. It is better to take a more balanced approach, and induce a reasonable amount of effort from both groups.

Next, consider more general objective functions, but assume that they are quasiconcave and symmetric in actions. A natural example arises when the designer seeks to maximize expected total production, in which case expected utility is

$$U_0(\mathbf{a}) = n_1\mathbb{E}[q_1|a_1] + n_2\mathbb{E}[q_2|a_2],$$

given group symmetry. The CDFC implies that $\mathbb{E}[q_i|a_i]$ is concave in a_i. Moreover, $\mathbb{E}[q_i|a_i]$ is symmetric since technologies are symmetric across groups.

When the designer's utility is quasiconcave, she cares less for the relatively extreme action profile that is implemented with $\kappa_1 = \kappa_1^{\Sigma}$. A more balanced action profile is preferable, and the optimal κ_1 moves closer to κ_1^s. Thus, even if $\kappa_1^{\Sigma} > 1$, the optimal κ_1 may fall below 1. For instance, this occurs in the limit case when the designer has CES preferences over $\mathbf{a}$ that approach Leontief preferences, leading the optimal κ_1 to approach κ_1^s. In other words, the degree to which actions are substitutes or complements is important in determining the optimal action profile and which group of agents win more often.

Proposition 8 *Assume $U_0(\mathbf{a})$ is differentiable, monotonic, quasi-concave, and* *symmetric. Assume that $v_1 > v_2$ and $t_i(a_i) = t(a_i)$, $i = 1, 2$, with $t(a_i)$ convex. Let κ_1^U denote the unique optimal value of κ_1. Then, $\kappa_1^U \in (\kappa_1^s, \kappa_1^\Sigma]$ and $a_1^* > a_2^*$ in equilibrium. Finally, even if $\kappa_1^\Sigma \geq 1$, there are eligible $U_0(\mathbf{a})$ for which $\kappa_1^U < 1$.*

Example 4 (Objective functions and optimal design): The designer's objective is to maximize total production. Assume that $H_i(q_i) = q_i^\gamma$, $q_i \in [0, 1]$, with $\gamma > 0$, and that $f_i(a_i) = c_i(a_i) = a_i$, $i = 1, 2$. Then, $\mathbb{E}[q_i|a_i] = \frac{\gamma a_i}{1+\gamma a_i}$. An increase in γ leads to a concave transformation of $\mathbb{E}[q_i|a_i]$. Thus, the designer is more interested in a balanced action profile the higher γ is, and the optimal design is therefore sensitive to the properties of $H_i(q_i)$.[11] It can be verified that $\kappa_1^U < 1$ if and only if $v_1 v_2 \gamma^2 > \frac{n^4}{(n-1)^2}$. This condition is more easily satisfied the higher v_1, v_2, and γ are, and the lower n is. Increases in valuations amplify actions and make it more important to smooth out actions across groups. Conversely, more agents leads to a more competitive contest, which on its own tends to smooth out actions across groups when κ_1 is close to 1. $\square$

Example 5 (Non-convexities): Consider a contest with two completely symmetric agents, such that $n_1 = n_2 = 1$ and $v_1 = v_2 = v$. Assume that $f_i(a_i) = a_i$ and that

$$c_i(a_i) = \begin{cases} \frac{1}{2}\gamma(\gamma - 1) a_i^2 - a_i^\gamma & \text{if } a_i \in [0, 1] \\ \frac{1}{2}(\gamma - 2)(2\gamma a_i - \gamma + 1) & \text{if } a_i \in (1, \infty) \end{cases},$$

for some $\gamma > 2$. The cost function is increasing and convex. The third derivative is sometimes strictly negative, sufficiently so to make $t_i(a_i)$ "very" concave locally. It can be shown that for *any* $\gamma > 2$, there exists a v value for which *the feasible set is non-convex.*[12] Figure 2 illustrates, assuming $\gamma = 10$ and $v = 300$. The iso-profit line along which total effort is maximized is also shown. Although agents are symmetric, treating them asymmetrically increases total effort by about 2.5%.

It follows that if valuations are perturbed slightly, such that $v_1 \neq v_2$, then

[11] In contrast, the frontier of the feasible set in Proposition 5 does not depend on H_1 and H_2. The designer can always transform q_i into the quantile $\widetilde{q}_i = H_i(q_i)$ and use this as the basis for contest design. It follows that the set of implementable action profiles is independent of H_1 and H_2. It is for the same reason that it is not required that $H_1 = H_2$ in Proposition 5.

[12] Here, c_i, c_i', c_i'' are all continuous. The linear extension for $a_i \geq 1$ just ensures that c_i is globally increasing. The proof of the assertion relies on the behavior of c_i for $a_i < 1$. The important property is that there are $a_i < 1$ values for which t_i is "so concave" that $t_i' + 2t_i t_i'' < 0$. Then, the MRT is *increasing* near the point where $a_1 = a_2$.

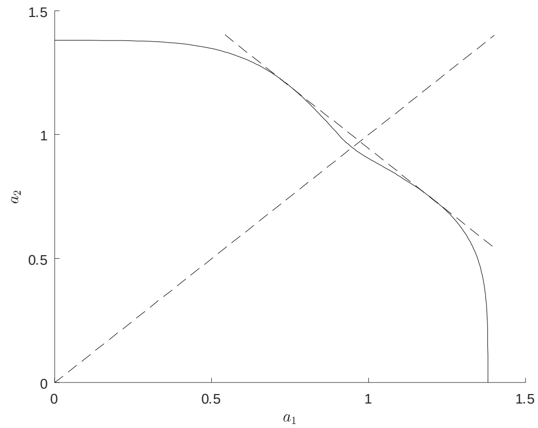

Figure 2: Non-convex feasible set

favoring the weaker agent lead to higher aggregate effort than levelling the playing field does, even if favoring the strong agent may be even better. Likewise, a small perturbation of the designer's preferences may cause a discontinuous change in the optimal design. □

Finally, it is straightforward to permit rationing. See Appendix C for details. This causes $F(\kappa_i|n_i, n_j)$ to increase and the feasible set to expand. Lemma 1 is unchanged, which means that the benchmark in (13) is the same as well. Thus, the main results are robust, although the size of κ_1^s may change. Given the conditions in Proposition 8, it remains the case that the strong group of agents are incentivized to work harder than the weak group, or $a_1^* > a_2^*$. As noted in Example 1, this implies that the stronger group faces a more stringent minimum standard, or $\widehat{q}_1(a_1^*) > \widehat{q}_2(a_2^*)$. The most significant qualitative change of allowing rationing is that Corollary 2 also holds when $n_i = 1$. This is another indication that the no-rationing case with $n_i = 1$ is a special case.

5.3 Contest success functions

Fullerton and McAfee (1999) work with a specialized version of the best-shot model, with $H_i(q_i) = H(q_i)$ for all $i \in N$. In words, all ideas are equally good ex ante but some agents may have more ideas than others. Then, agent i wins an unbiased contest with probability

$$p_i(\mathbf{a}) = \frac{f_i(a_i)}{\sum_{j=1}^n f_j(a_j)}, \tag{14}$$

when $\sum_{j=1}^n f_j(a_j) > 0$. This is intuitive because each idea has an equal chance of being the best. Hence, the Fullerton and McAfee (1999) model provides a microfoundation for the *unbiased* lottery CSF. The question now is how to proceed to model and analyze biased contests.

The premise of the current paper is that performance is observed and that this can be used as the basis for contest design. In the best-shot model, τ_1 and τ_2 determine the equilibrium action profile and thus the scoring functions in (11). With these in hand, the implied CSF can be derived by carrying out the integration in (2). Appendix C demonstrates that the resulting CSF is *not* a lottery CSF – it does not take a simple ratio form as in (14). Thus, even though the starting point is an unbiased lottery contest, endogenizing the contest design fundamentally alters the resulting CSF.[13]

This point clashes with a popular approach in the existing contest literature. It is common to assume that the designer can implement a CSF that is some variant of

$$p_i(\mathbf{a}|\boldsymbol{\delta}, \mathbf{b}, z) = \frac{b_i f_i(a_i) + \delta_i}{\sum_{j=1}^n (b_j f_j(a_j) + \delta_j) + z}. \tag{15}$$

Here, one or more of $\boldsymbol{\delta} = (\delta_1, ..., \delta_n)$, $\mathbf{b} = (b_1, ..., b_n)$, and z are considered to be design instruments. Typically, each variable is restricted to be non-negative. The interpretation of δ_i is that it captures agent i's head start, while b_i is a multiplicative bias or handicap. Thus, it is as if agent i earns $b_i f_i(a_i) + \delta_i$ lottery tickets. Finally, z can be thought of as the number of lottery tickets that the designer reserves for herself, thus admitting the possibility of rationing.

It is hard to reconcile (15) with the premise in Fullerton and McAfee (1999) that the qualities of ideas are stochastic, at least as long as actions are taken to be unobservable: If actions are unobservable, then how are biases and head starts

[13] There are also axiomatic justifications for the lottery CSF, see Skaperdas (1996) and Clark and Riis (1998). However, once contest design is endogenized, there appear little reason to think that the designer will voluntarily limit herself to contests that satisfy nice axioms.

applied to $f_i(a_i)$ in order to calculate the new number of lottery tickets?[14]

Methodologically, the problem is that (15) appears to treat the CSF as the primitive. However, in Fullerton and McAfee (1999) it is the performance technology G_i that is the primitive. The CSF is just a reduced form that integrates out the uncertainty over q.[15] Given that the unbiased CSF is obtained from the stochastic performance premise, internal consistency demands that the stochastic performance premise remains the foundation for biased contests as well. This leads to the approach taken in this paper.

It is also worth emphasizing how hard it is to draw practical implications from the biased lottery approach. Again, how should one go about levelling the playing field by ensuring that each agent has an equal number of lottery tickets when this number is not observable? In other words, it is hard to see how policy recommendations should be implemented in practice since all results and predictions relate to something that is unobservable. The approach in the current paper instead allows the observable variables to take centre stage since the assignment rule is explicitly based on the observed performance profile.

The following example contrasts the two approaches.

Example 6 (Head starts and Handicaps in AIM contests): An active literature relies on (15) to derive optimal head starts and handicaps, usually with the assumption that $z = 0$.[16] For concreteness, assume that $n = 2$ and that $f_i(a_i) = c_i(a_i) = a_i$, $i = 1, 2$. Using (15), Fu and Wu (2020) show that individual actions are maximized by perfectly levelling the playing field such that each agent wins with probability 0.5.[17] Since individual actions are maximized, this design

[14] If actions are observable, an auction-like mechanism is likely preferable to a lottery contest. There is no reason to think that the ad hoc CSF in (15) would be optimal.

[15] Another way of expressing the problem is that the literature has not provided a microfoundation for (15). Appendix C provides such a microfoundation. This is again based on transforming performance into scores. The transformation in question is ad hoc, but it can be done. Since the transformation is ad hoc, the resulting assignment rule is suboptimal. Hence, (15) underestimates the value of optimal contest design. See Example 6.

[16] Dasgupta and Nti (1998) consider a model with symmetric agents. The designer's own-use valuation is $v_0 \geq 0$ and she benefits from total effort. This is equivalent to a contest with a single prize that is costly to the designer. Dasgupta and Nti (1998) rely on (15) and show that $z = 0$ (no rationing) is optimal when v_0 is sufficiently small. In contrast, Proposition 4 proves that rationing is always optimal when using the stochastic performance approach.

[17] For other recent papers in this literature, see Franke (2012), Franke, Leininger, and Schwartz (2013), and Franke, Leininger, and Wasser (2018).

is optimal whenever the designer's preferences are strictly monotonic. In the ensuing equilibrium, $a_i = \frac{v_i}{4}$, $i = 1, 2$, and $a_1 + a_2 = \frac{v_1+v_2}{4}$ or $a_1 + a_2 = \frac{k+1}{4} v_2$, where $k = \frac{v_1}{v_2} > 1$ is a measure of the asymmetry. Incidentally, the same outcome is obtained in Proposition 5 by letting $\kappa_1 = 1$.

However, if the objective is to maximize total effort, (13) establishes that the optimal design involves $\kappa_1 = \frac{v_1}{v_2} = k$, which yields $a_1 + a_2 = \frac{k^2}{k+1} e^{\frac{1}{k}-1} v_2$. The percentage improvement is increasing in k, and converges to $\frac{4-e}{e}$ – or just above 47% – as $k \to \infty$. Hence, using the fully optimal and microfounded design can lead to a substantial increase in payoff. Welfare implications are also different, as it is no longer optimal to fully level the playing field. □

Remark 1 (The exponential noise model): The working paper version, Kirkegaard (2020), contains a detailed discussion of the exponential noise model by Hirschleifer and Riley (1992) that also justifies the lottery CSF when $n = 2$. Although the CDFC does not apply, it is confirmed that the paper's approach still works when $n = 2$ and $c_i(a_i) = f_i(a_i) = a_i$, $i = 1, 2$. Indeed, in this case, the feasible set exactly coincides with the feasible set of the best-shot model. The optimal action profile in the two models is therefore the same, as long as $U_0(\mathbf{a})$ is the same. However, agent i's winning probability is *decreasing* in κ_i in the Hirschleifer and Riley (1992) model. Thus, the two models yield the exact opposite conclusions in terms of which agent wins more often under the optimal design. This fits Fu and Lu's (2012) observation that the exponential noise model is essentially a "worst-shot" model and therefore in many ways the opposite of the best-shot model. Indeed, with reference to Corollary 1, $G_L(L|a)$ is concave in L in the exponential noise model, but convex in the best-shot model □

6 Discussion

6.1 Inferences from data

It has been assumed that the designer knows the primitives. If the performance of a large number of unbiased contests with the same sort of contestants have been observed, then $G_i(q_i|a_i^*)$ can be estimated econometrically even though the equilibrium action a_i^* is unknown. However, this is typically not sufficient to infer

LRs.[18] Nevertheless, past contest data may be valuable in some circumstances, such as when the "parametric family" of the distribution is known.

Example 7 (Inferences in the best-shot model): Consider the best-shot model with a single prize, but without any prior knowledge about H_i, f_i, and c_i. In the unbiased contest, the first-order conditions are

$$v_i \int_{\underline{q}_i}^{\overline{q}_i} G_j(q_j|a_j^*)^{n_j} G_i(q_i|a_i^*)^{n_i-1} L_i(q_i|a_i^*) g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0$$

which in the best-shot model simplifies to

$$v_i \int_{\underline{q}_i}^{\overline{q}_i} G_j(q_j|a_j^*)^{n_j} G_i(q_i|a_i^*)^{n_i-1} \left(1 + \ln G_i(q_i|a_i^*)\right) g_i(q_i|a_i^*) dq_i = t_i(a_i^*).$$

Assuming v_i is known, the left hand side depends only on known or observable terms. Thus, $t_i(a_i^*)$ can be inferred. Even though $G_i(q_i|a_i^*)$ and $t_i(a_i^*)$ is only "local" information (at $a_i = a_i^*$), it may still hold value. Assume that H_i, f_i, and c_i are heterogenous across two groups but that the prize is a sum of money, v, that is valued the same across groups. The designer can now lower v while still maintaining the same effort. The reason is that the original action profile is not on the frontier of the feasible set that is implied by v. By lowering v, the feasible set shrinks, and the action profile is eventually on the frontier. Thus, the same action profile can be implemented by lowering v to $\widehat{v}$ until the system

$$t_1(a_1^*) = \widehat{v} F\left(\kappa_1|n_1, n_2\right) \text{ and } t_2(a_2^*) = \widehat{v} F\left(1/\kappa_1|n_2, n_1\right),$$

has a solution for some κ_1.

Consider next a contest with a non-monetary prize but assume that valuations are known and that it is known that $H_i = H$ for all $i \in N$. Then, information about f_i comes from observed winning frequencies, via (14). If the parametric forms of c_i and f_i are also known, then it may be possible to go even further. For

[18] Even if a_i^* is observed ex post, this is not of that much value as $G_i(q_i|a_i^*)$ is still only a "point estimate". Similarly, winning frequencies at a fixed equilibrium action profile do not in general identify the CSF away from the equilibrium action profile. Indeed, even if the unbiased CSF is known, Section 5.3 makes the point that it is unclear how to use this for design purposes.

instance, assume $c_i(a_i) = a_i$ and $f_i(a_i) = a_i^r$, where $r \in (0, 1]$ is unknown. Since $\frac{f_i(a_i^*)}{f_j(a_j^*)} = \left(\frac{a_i^*}{a_j^*}\right)^r$ is known (from winning frequencies) and $\frac{t_i(a_i^*)}{t_j(a_j^*)} = \frac{a_i^*}{a_j^*}$ is known (from first-order conditions), r can be inferred. Then, a_i^* and a_j^* follow from the first-order conditions. Knowing $f_i(a_i)$, it is also possible to infer H from the observed distribution $G_i(q_i|a_i^*)$, and thus construct $G_i(q_i|a_i)$ for all a_i. The designer now has all she needs for optimal design. $\square$

6.2 Extensions to the model

It is possible to generalize the model beyond the rationing/no rationing dichotomy. Restrictions on Ω may reflect the designer's inability to commit to, or legal roadblocks that preclude, certain assignments. For instance, a law that a specific percentage of prizes must be awarded to women or minorities would imply that not all constellations of winners are feasible.

In the general case, each feasible assignment, $\omega \in \Omega$, is evaluated by an aggregate score, much as in Proposition 4. This may change the assignment even in AIM contests (Theorem 1), because the assignment that awards prizes to the agents with the highest individual scores may not be feasible. For details, see the working paper, Kirkegaard (2020).

The general logic behind optimal contest design remains the same when there are more than two groups of agents or when identical agents do not have to be treated symmetrically. However, it is more cumbersome to describe the frontier of the feasible set. Again, see Kirkegaard (2020) for details.

6.3 Scoring rules, prize splitting, and entry fees

Lazear and Rosen (1981) consider an extension to tournaments in which one agent is given a head start. See also Fain (2009). Then, each agent has a linear scoring rule. However, since LRs are rarely linear in q_i, this is quite unlikely to be optimal. Indeed, even when LRs are linear, the scoring functions are unlikely to have the same slope. Jewitt (1988) argues that the LR is often concave in performance. This is the case in the best-shot model when H_i is log-concave.

Similarly, Nalebuff and Stiglitz (1983) and Imhof and Kräkel (2006) allow the prize to be split among agents if performances are close. Since this is not a deterministic assignment rule, such a policy is not optimal in the current setting.

Finally, entry fees can be added to the model. Optimal entry fees are identity-dependent and force the participation constraints to bind. For AIM contests with a single and common monetary prize of value v, the size of v can likewise be endogenized as in Example 7. Increasing the prize expands the feasible set. Given quasilinear preferences as well as endogenous entry fees and prizes, the first-best can be achieved in AIM contests. Starting from $v = 0$, simply expand the feasible set enough to implement the first-best solution to $U_0(\mathbf{a}) - \sum_{i \in N} c_i(a_i)$ and then extract all rent via entry fees.

7 Conclusion

This paper pursues a model of contests that is based on stochastic performance. Contest design is then a team moral hazard problem in which the assignment rule is manipulated to incentivize effort. The principles behind optimal design are remarkably robust to both the designer's objectives and the distributions of performance. Consistent with the standard single-agent principal-agent model, likelihood-ratios play a key role in determining an agent's compensation or, in this case, the probability that he wins a prize. Nevertheless, the specifics of the optimal design and the induced action profile depend on the designer's objective function and primitives like the distribution of performance and the cost functions. For instance, which group of agents is favored with higher winning probabilities depends on all these factors.

The model provides both practical and conceptual insights. In practical terms, it endogenizes standards for eligibility and explains why the number of prizes that are awarded may be stochastic ex ante. Similarly, the optimal design features a threshold of excellence beyond which advantaged agents are insulated from competition from other groups. Conceptually, the approach offers an alternative to the literature that is based on manipulating a black-box CSF. The current approach instead bases design on observable signals.

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Appendix A: Proofs of main results

Proof of Proposition 1. The proof characterizes the set of implementable a_i and proves the assertion in Proposition 1. Let $\widehat{q}_i(a_i)$ denote the unique value of q_i for which $L_i(q_i|a_i) = 0$. Now fix some target action, a_i^t, that the designer may wish to implement. As explained at the beginning of Section 3.2, when evaluated at $a_i = a_i^t$, (3) is maximized with a threshold assignment rule that has the property that

$$P_i(q_i, \mathbf{q}_{-i}) = \begin{cases} 1 & \text{if } q_i \geq \widehat{q}_i(a_i^t) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

The threshold rule in (16) is useful because by construction it maximizes the first derivative in (3) when evaluated at $a_i = a_i^t$. Hence, if the threshold rule leads (3) to take a negative value, then there is no assignment rule that can feasible satisfy the first-order condition. Then, the target action a_i^t simply cannot be implemented. Hence, it is necessary for implementability that (3) is non-negative at a_i^t when the threshold rule is used. This is a sufficient condition as well. To see this, consider a threshold rule with threshold $q_i = \underline{q}_i$. Then, the agent never wins, regardless of his performance. Hence, (3) is strictly negative at $a_i = a_i^t$. By continuity, there must then exist some threshold between $\underline{q}_i$ and $\widehat{q}_i(a_i^t)$ for which (3) is exactly zero when evaluated at $a_i = a_i^t$. Since this threshold rule is monotonic, the agent's expected utility is concave by the CDFC and the first-order condition is thus sufficient.

More precisely, given (16), agent i's expected utility from some action a_i is

$$v_i(1 - G_i(\widehat{q}_i(a_i^t)|a_i)) - c_i(a_i). \tag{17}$$

Hence, following the above argument, a_i^t is implementable if and only if

$$-\left.\frac{\partial G_i(\widehat{q}_i(a_i^t)|a_i)}{\partial a_i}\right|_{a_i=a_i^t} \geq \frac{c_i'(a_i)}{v_i}. \tag{18}$$

The MLRP implies that the left-hand side is strictly positive.

Moreover, the $P_i(q_i, \mathbf{q}_{-i})$ function that implements a_i is (essentially) unique if and only if (18) is binding. First, $P_i(q_i, \mathbf{q}_{-i})$ is not unique when (18) is slack.

It has already been established that there is a threshold rule that implements such an a_i. However, by similar reasoning, there is another threshold rule with threshold above $\widehat{q}_i(a_i^t)$ that satisfies the first-order condition. For any action for which (18) binds, the assignment rule is essentially unique in its description of $P_i(q_i, \mathbf{q}_{-i})$ because the threshold rule maximizes (3). Thus, any assignment rule that differs on a set of performances profiles of positive measure would fail to satisfy the agent's first-order condition. This last part proves the proposition. ■

Proof of Theorem 1. The task is to identify action profiles that are group-symmetric and on the frontier of the feasible set. The corners are by definition where agents in group i take action $\underline{a}_i^s$ and agents in group j take action $\overline{a}_j^s$, $i \neq j$, and $i, j = 1, 2$. This proposition describes the rest of the frontier. Here, both groups must take actions strictly higher than $\underline{a}_i^s$, $i = 1, 2$. Otherwise, the other group j can be induced to take action $\overline{a}_j^s$, but this describes either a corner (if $a_i = \underline{a}_i^s$) or a point on the boundary that is not on the frontier (if $a_i < \underline{a}_i^s$). Hence, the actions of agents in group i is in $(\underline{a}_i^s, \overline{a}_i^s)$, $i = 1, 2$. Since actions are interior, incentive compatibility necessitates that the agents' first-order conditions are satisfied. The idea is to use the first-order approach by assuming (and then verifying) that the first-order conditions are also sufficient.

Ignoring group-symmetry to start, any action profile $\mathbf{a} = (a_j, \mathbf{a}_{-j})$ that is on the frontier must have the property that a_j is maximized given $\mathbf{a}_{-j}$. For a fixed j and $\mathbf{a}_{-j}$, the assignment rule must therefore solve

$$\max_{a_j, \{P_\omega(\mathbf{q})\}_{\omega \in \Omega, \mathbf{q} \in Q}} a_j \tag{19}$$

$$st \quad \frac{\partial U_i(a_i, \mathbf{a}_{-i} | \mathbf{P})}{\partial a_i} = 0, \quad \text{for all } i \in N$$

$$P_\omega(\mathbf{q}) \geq 0, \quad \text{for all } \mathbf{q} \in Q \text{ and all } \omega \in \Omega$$

$$\sum\nolimits_{\omega \in \Omega} P_\omega(\mathbf{q}) = 1, \quad \text{for all } \mathbf{q} \in Q,$$

where $Q = \times_{i \in N} [\underline{q}_i, \overline{q}_i]$. Combining (1) and (3) means that the first set of constraints can be written

$$\int \left(\sum\nolimits_{\{\omega \in \Omega | i \in \omega\}} P_\omega(\mathbf{q}) v_i L_i(q_i | a_i) \right) \prod\nolimits_{k \in N} g_k(q_k | a_k) d\mathbf{q} - c_i'(a_i) = 0 \text{ for all } i \in N$$

or

$$\mathbb{E}\left[\sum\nolimits_{\{\omega\in\Omega|i\in\omega\}} P_\omega(\mathbf{q})v_i L_i(q_i|a_i)|\mathbf{a}\right] - c_i'(a_i) = 0.$$

It is convenient to write the second and third sets of constraints as

$$\begin{aligned} P_\omega(\mathbf{q})\prod\nolimits_{k\in N} g_k(q_k|a_k) &\geq 0, \quad \text{for all } \mathbf{q}\in Q \text{ and all } \omega\in\Omega \\ \left(\sum\nolimits_{\omega\in\Omega} P_\omega(\mathbf{q}) - 1\right)\prod\nolimits_{k\in N} g_k(q_k|a_k) &= 0, \quad \text{for all } \mathbf{q}\in Q. \end{aligned}$$

Let $\{\mu_i\}_{i\in N}$ denote the multipliers to the first set of constraints and$\{\lambda_\omega(\mathbf{q})\}_{\omega\in\Omega,\ \mathbf{q}\in Q}$ and $\{\eta(\mathbf{q})\}_{\mathbf{q}\in Q}$ the multipliers to the second and third set of constraints, respectively. The Lagrangian can then be written as

$$a_j + \mathbb{E}\left[\sum_{i\in N}\sum_{\{\omega\in\Omega|i\in\omega\}} P_\omega(\mathbf{q})\mu_i v_i L_i(q_i|a_i) + \sum_{\omega\in\Omega}\lambda_\omega(\mathbf{q})P_\omega(\mathbf{q}) + \eta(\mathbf{q})\left(\sum_{\omega\in\Omega} P_\omega(\mathbf{q}) - 1\right)|\mathbf{a}\right] - \sum_{i\in N}\mu_i c_i'(a_i)$$

For a given assignment ω and a given performance profile $\mathbf{q}$, the first-order condition with respect to $P_\omega(\mathbf{q})$ is

$$\sum_{i\in\omega}\mu_i v_i L_i(q_i|a_i) + \lambda_\omega(\mathbf{q}) = -\eta(\mathbf{q}),$$

where the right hand side is independent of ω. Hence, $\lambda_\omega(\mathbf{q})$ is smallest for the assignment ω which maximizes the first term on the left hand side. Since $\lambda_\omega(\mathbf{q}) \geq 0$, this means that $\lambda_\omega(\mathbf{q}) > 0$ for all ω that do not maximize this first term. Thus, $P_\omega(\mathbf{q}) = 0$ for such assignments. Hence, feasibility dictates that

$$\sum\nolimits_{\omega\in\Omega(\mathbf{q},\boldsymbol{\mu}|\mathbf{a})} P_\omega(\mathbf{q}) = 1$$

where

$$\Omega(\mathbf{q},\boldsymbol{\mu}|\mathbf{a}) = \{\omega\in\Omega|\sum\nolimits_{i\in\omega}\mu_i v_i L_i(q_i|a_i) \geq \sum\nolimits_{i\in\omega'}\mu_i v_i L_i(q_i|a_i) \text{ for all } \omega'\in\Omega\}.$$

This assignment rule is implemented by assigning prizes to the agents with the highest scores, as described in the statement of the theorem.

Next, it is necessary to sign $\{\mu_i\}_{i\in N}$. To begin, since agent i is incentivized to take a positive action he must win a prize with strictly positive probability. Then, it is easy to rule out that $\mu_i < 0$. In this case, by the MLRP, agent i's score diminishes when q_i increases, meaning that any assignment ω that he is a member of gets a lower aggregate score. Thus, any such assignment is less likely to be implemented. Stated differently, $P_i(q_i, \mathbf{q}_{-i})$ is decreasing in q_i if $\mu_i < 0$. This violates the incentive constraint in the maximization problem as the agent then has an incentive to deviate downwards. Hence, $\mu_i \geq 0$. The difficulty is in ruling out that $\mu_i = 0$. To this end, it is useful to consider the first-order condition for a_j in (19), which is

$$1 + \sum_{i \in N} \mu_i \frac{\partial^2 U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i \partial a_j} = 0. \tag{20}$$

It follows that μ_i cannot be zero for all $i \in N$. In other words, there is some agent $i \in N$ with $\mu_i > 0$. The aim is to show that $\mu_i > 0$ for all $i \in N$, or to rule out that $\mu_i = 0$ for any $i \in N$. Now, there is a problem like (19) for any $j \in N$. The equilibrium assignment rule must solve all these problems, or the action profile would not be on the frontier. Thus, regardless of which $j \in N$ is considered in (19), the same μ_i multipliers must solve the problem. By extension, (20) holds for all $j \in N$.

Now assume by contradiction that $\mu_j = 0$ for some agent $j \in N$. Consider how this latter agent j interacts with any agent i for which $\mu_i > 0$. Since $\mu_j = 0$, agent j's score is $\mu_j v_j L_j(q_j|a_j) = 0$ regardless of q_j. Thus, $\Omega(\mathbf{q}, \boldsymbol{\mu}|\mathbf{a})$ is independent of q_j. Therefore, q_j does not matter from agent i's point of view unless possibly if there are distinct assignments ω and ω' in $\Omega(\mathbf{q}, \boldsymbol{\mu}|\mathbf{a})$ such that agent i is a member of ω but not ω', in which case the value of q_j could be used as a tie-breaker to determine whether agent i receives a prize or not. However, this is a probability zero event. The reason is that $\mu_i > 0$ means that agent i's score is strictly increasing in q_i. Therefore, given $\mathbf{q}_{-i}$, the aggregate score of any assignment of which agent i is a member is strictly increasing in q_i.

Thus, q_j does not impact agent i. A marginal increase in a_j changes the distribution of q_j, but this is irrelevant to agent i. Hence, $\mu_i \frac{\partial^2 U_i(a_i, \mathbf{a}_{-i}|\mathbf{P})}{\partial a_i \partial a_j} = 0$ if $\mu_i > 0$ and $\mu_j = 0$. Thus, all the terms under the summation sign in (20) are zero, which means that (20) is violated. It follows that $\mu_j > 0$ for all $j \in N$.

Since the multipliers are positive, any agent obtains a strictly higher score the higher his performance is, by the MLRP. Thus, the probability that he is assigned a prize increases. In other words, $P_i(q_i, \mathbf{q}_{-i})$ is monotonic in q_i and the CDFC now implies that the agent's problem is concave. Hence, the first-order condition is sufficient. That is, the first-order approach is valid.

Thus far, group symmetry has not been invoked. Hence, the proof demonstrates a principle that extends to contests with more groups or in which group symmetry is not imposed. However, as discussed in the working paper version, Kirkegaard (2020), there are other complications in that case. Thus, the remainder of the proof makes use of the assumption that there are exactly two groups and that group symmetry is imposed. Assume that agents in group 1 must all be induced to take the same action, a_1. This means that two distinct members of group 1 must have multipliers that take the same value. Otherwise, the agent with the higher μ_i wins more often when his likelihood-ratio is positive and less often when his likelihood-ratio is negative than the agent with the lower multiplier does. However, this means that the former has stronger incentives than the latter on the margin, starting at the common action a_1. This violates the incentive constraint of at least one of the agents. Therefore, the multipliers must be group symmetric. ■

Proof of Proposition 2. By definition of NUC, $-\frac{\partial G_i(q_i|a_i)}{\partial a_i}$ is log-supermodular in a_i and q_i, or

$$\frac{\partial^2}{\partial q_i \partial a_i} \ln \left(-\frac{\partial G_i(q_i|a_i)}{\partial a_i} \right) \geq 0$$

which is equivalent to

$$\frac{\partial^2}{\partial a_i} \left(\frac{\partial g_i(q_i|a_i)}{\partial a_i} / \frac{\partial G_i(q_i|a_i)}{\partial a_i} \right) \geq 0.$$

Recall that the MLRP implies that $\frac{\partial G_i(q_i|a_i)}{\partial a_i}$ is negative and that $\frac{\partial g_i(q_i|a_i)}{\partial a_i}$ is negative when q_i is small and positive when q_i is large. Thus, the ratio of the two is first positive and then negative as q_i increases. The fact that the ratio is increasing in a_i then means that it is positive for more q_i when a_i is higher, or in other words that $\frac{\partial g_i(q_i|a_i)}{\partial a_i}$ is negative for more q_i. Since $\widehat{q}_i(a_i)$ is defined as the performance where $\frac{\partial g_i(q_i|a_i)}{\partial a_i}$ is zero, it follows that it is weakly increasing in a_i. ■

Proof of Proposition 3. Agent i's first-order condition, (3), can be written more concisely as

$$v_i \mathbb{E}[P_i(\mathbf{q}) L_i(q_i|a_i)|\mathbf{a}] - c_i'(a_i) = 0.$$

Since technologies are symmetric and costs are convex, it must hold that

$$v_1 \mathbb{E}[P_1(\mathbf{q}) L_1(q_1|a_1)|\mathbf{a}] = v_2 \mathbb{E}[P_2(\mathbf{q}) L_2(q_2|a_2)|\mathbf{a}]$$

whenever $a_2 = a_1$ is implemented. Since $v_1 > v_2$, this in turn requires that

$$\mathbb{E}[P_1(\mathbf{q}) L_1(q_1|a_1)|\mathbf{a}] < \mathbb{E}[P_2(\mathbf{q}) L_2(q_2|a_2)|\mathbf{a}],$$

If $\mu_1 v_1 = \mu_2 v_2$, then the scoring rule is symmetric and the left and right hand sides coincide. If $\mu_1 v_1 > \mu_2 v_2$ then $P_1(\mathbf{q})$ jumps from zero to one for some performance profiles where $L_1(q_1|a_1) > 0$ and – if rationing is ruled out – from one to zero for some performance profiles where $L_1(q_1|a_1) < 0$. The opposite occurs for agent 2. Hence, the left hand side increases and the right hand side decreases. In either case, the required inequality is violated. Thus, $\mu_2 v_2 > \mu_1 v_1$ is optimal.

If rationing is allowed, then $P_i(\mathbf{q}) = 0$ whenever $L_i(q_i|a_i) < 0$. Since only positive likelihood-ratios are ever rewarded, increasing $\mu_2 v_2$ above $\mu_1 v_1$ then unambiguously implies that agent 2 outscores agent 1 more often. Hence, agent 2 wins more often in expectation than agent 1. ■

Proof of Corollary 1. If $a_1 = a_2 = a^*$ is induced, then $\mu_2 v_2 > \mu_1 v_1$ from Proposition 3. Hence, agent 2's scores have a wider range. Agent 1 wins if and only if $L_2 < \alpha L_1$, where $\alpha = \frac{\mu_1 v_1}{\mu_2 v_2} \in (0, 1)$. Ex ante, agent 1 then wins with probability

$$\mathbb{E}[P_1(\mathbf{q})|\mathbf{a}^*] = \int_{L_1(\underline{q}_1|a^*)}^{L_1(\overline{q}_1|a^*)} G_L(\alpha L|a^*) g_L(L|a^*) dL,$$

and agent 2 wins with the remaining probability. Note that if $\alpha = 1$, then

$\mathbb{E}[P_1(\mathbf{q})|\mathbf{a}^*] = \frac{1}{2}$. The derivative with respect to α is

$$\int Lg_L(\alpha L|a^*)g_L(L|a^*)dL = -\int \left(\int_{L_1(\underline{q}_1|a^*)}^{L} xg_L(x|a^*)dx \right) \frac{\partial g_L(\alpha L|a^*)}{\partial L} dL,$$

where integration by parts was used, along with the fact that the LR is zero in expectation. The latter fact also implies that the inner integral on the right is negative. Since $\alpha < 1$, this means that agent 1 wins less often than agent 2 if $G_L(L_i|a^*)$ is convex in L_i, but the opposite holds if $G_L(L_i|a^*)$ is concave in L_i. This completes the proof. ■

Proof of Proposition 4. The proof follows the same steps as the proof of Theorem 1, but modified to account for the designer's more general preferences. Given assignment rule $\mathbf{P}$ and action profile $\mathbf{a}$, the designer's expected utility is

$$\begin{aligned} U_v(\mathbf{a}|\mathbf{P}) &= \mathbb{E}[\textstyle\sum_{\omega\in\Omega} \pi_\omega(\mathbf{q},\mathbf{a})P_\omega(\mathbf{q})|\mathbf{a}] \\ &= U_0(\mathbf{a}) + \mathbb{E}[\textstyle\sum_{\omega\in\Omega} \upsilon_\omega(\mathbf{a})P_\omega(\mathbf{q})|\mathbf{a}] \\ &= \mathbb{E}[\textstyle\sum_{\omega\in\Omega} \left(U_0(\mathbf{a}) + \upsilon_\omega(\mathbf{a})\right) P_\omega(\mathbf{q})|\mathbf{a}], \end{aligned}$$

where $U_0(\mathbf{a})$ is again the expected value of $\pi(\mathbf{q},\mathbf{a})$ and where the last equality follows from the fact that probabilities sum to one for all performance profiles. The objective is to maximize $U_v(\mathbf{a}|\mathbf{P})$ subject to the same feasibility constraints as in the proof of Theorem 1. The same arguments then establish that the score of any assignment is

$$\pi_\omega(\mathbf{q},\mathbf{a}) + \textstyle\sum_{i\in\omega} \mu_i \upsilon_i L_i(q_i|a_i),$$

but since $\pi(\mathbf{q},\mathbf{a})$ cancel out, scores can instead be computed as

$$\upsilon_\omega(\mathbf{a}) + \textstyle\sum_{i\in\omega} \mu_i \upsilon_i L_i(q_i|a_i).$$

The optimal assignment rule must assign probability one among the assignments with the highest scores. This produces the rule in Proposition 4.

To sign the multipliers, consider the first-order condition for a_j in the maximization problem,

$$\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j} + \sum_{i\in N} \mu_i \frac{\partial^2 U_i(a_i, \mathbf{a}_{-i}|\mathbf{P}^*)}{\partial a_i \partial a_j} = 0, \tag{21}$$

where $\mathbf{P}^*$ is an optimal assignment rule. The first step is to show that $\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j} > 0$ if $\mu_j = 0$. To this end, assume that $\mu_j = 0$ and note that q_j as a consequence does not impact the score of any assignment. Thus, let $\Omega_{\mathbf{q}_{-j}}$ denote the set of assignments with the highest scores, given $\mathbf{q}_{-j}$. This may have several elements, but even in this case the value of $v_\omega(\mathbf{a})$ is the same for all $\omega \in \Omega_{\mathbf{q}_{-j}}$ for almost all $\mathbf{q}_{-j}$.[19] Now write $U_v(\mathbf{a}|\mathbf{P}^*)$ as

$$U_v(\mathbf{a}|\mathbf{P}^*) = \int \left(\int \sum_{\omega\in\Omega_{\mathbf{q}_{-j}}} \left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right) P^*_\omega(\mathbf{q}) g_j(q_j|a_j) dq_j \right) \prod_{i\neq j} g_i(q_i|a_i) d\mathbf{q}_{-j}.$$

Then, $\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j}$ is determined by the derivative of the inner integral, which for a fixed $\mathbf{q}_{-j}$ is

$$\int \left(\sum_{\omega\in\Omega_{\mathbf{q}_{-j}}} \frac{\partial \left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right)}{\partial a_j} P^*_\omega(\mathbf{q}) g_j(q_j|a_j) dq_j \right)$$
$$+ \int \left(\sum_{\omega\in\Omega_{\mathbf{q}_{-j}}} \left(U_0(\mathbf{a}) + v_\omega(\mathbf{a})\right) P^*_\omega(\mathbf{q}) L_j(q_j|a_j) g_j(q_j|a_j) dq_j \right)$$

By assumption, $a_j > 0$. This necessitates that agent j has a strictly positive probability of winning under $\mathbf{P}^*$. Hence, the first line is strictly positive for a set of $\mathbf{q}_{-j}$ of positive measure, by (9). Turning to the second line, for almost all $\mathbf{q}_{-j}$, the value of $U_0(\mathbf{a}) + v_\omega(\mathbf{a})$ across $\Omega_{\mathbf{q}_{-j}}$ is, as explained above, unique. Hence, the fact that a change in a_j changes the distribution of q_j and with it potentially the choice of assignment in $\Omega_{\mathbf{q}_{-j}}$ has no impact almost always. Hence, the expectation of the first line is strictly positive, while the expectation of the second line is zero. Therefore, $\frac{\partial U_v(\mathbf{a}|\mathbf{P}^*)}{\partial a_j} > 0$ if $\mu_j = 0$. Thus, the first term in (21) is strictly positive and the same arguments as in the proof of Theorem 1 can then be used again to

[19] For example, if $\mu_i = 0$ and $a_i = a$ for all agents, then any assignment that allocates all prizes yields the same score and the same value of $v_\omega(\mathbf{a})$ if $v_\omega(\mathbf{a}) = \sum_{i\in\omega} a_i$. However, more generally, two assignments ω and ω' with $v_\omega(\mathbf{a}) \neq v_{\omega'}(\mathbf{a})$ obtain the same score only if $\sum_{i\in\omega} \mu_i v_i L_i(q_i|a_i) - \sum_{i\in\omega'} \mu_i v_i L_i(q_i|a_i) = v_{\omega'}(\mathbf{a}) - v_\omega(\mathbf{a}) \neq 0$, which occurs with probability zero.

complete the proof of Proposition 4. ■

Proof of Proposition 5. See Proposition 11 in Appendix C. For completeness, Corollary 5 in Appendix C derives the action profiles that are at the corners of the frontier of the feasible set. ■

Proof of Lemma 1. Given the description of the frontier of the feasible set, note that

$$t_1'(a_1^*)da_1^* = v_1 F'\left(\kappa_1|n_1, n_2\right) d\kappa \text{ and } t_2'(a_2^*)da_2^* = v_2 F'\left(\frac{1}{\kappa_1}|n_2, n_1\right)\frac{-1}{\kappa_1^2}d\kappa.$$

It follows that the slope of the frontier is

$$\frac{da_2^*}{da_1^*} = -\frac{t_1'(a_1^*)}{t_2'(a_2^*)}\frac{v_2}{v_1}\frac{F'\left(\frac{1}{\kappa_1}|n_2, n_1\right)\frac{1}{\kappa_1^2}}{F'\left(\kappa_1|n_1, n_2\right)}.$$

The last factor can be simplified, yielding

$$\frac{da_2^*}{da_1^*} = -\frac{t_1'(a_1^*)}{t_2'(a_2^*)}\frac{v_2}{v_1}\frac{n_1}{n_2}\kappa_1.$$

Next, recall that a_1^* is increasing in κ_1 and that a_2^* is decreasing in κ_1. It now follows that as κ_1 increases, the ratio $\frac{t_1'(a_1^*)}{t_2'(a_2^*)}$ increases as well whenever $t_1(a_1)$ and $t_2(a_2)$ are convex functions. Hence, the slope decreases and becomes "more negative". In other words, the curve that describes the frontier of the feasible set is concave, as in Figure 1. Hence, the feasible set is convex. ■

Proof of Proposition 6. It can be shown that an agent in group i wins with probability

$$W\left(\kappa_i|n_i, n_j\right) = \begin{cases} e^{n_j(\kappa_i - 1)}\frac{1}{n_j\kappa_i + n_i} & \text{if } \kappa_i \in (0,1) \\ \frac{1}{n_i}\left(1 - n_j e^{n_i\left(\frac{1}{\kappa_i}-1\right)}\frac{\kappa_i}{n_j\kappa_i + n_i}\right) & \text{if } \kappa_i \geq 1 \end{cases}$$

along the frontier of the feasible set. There are a few ways of proving this, so here one is chosen that highlights an interesting feature of the distribution of scores.

It follows from (11) that the equilibrium distribution of agent i's score is

$$K_i(s|\tau_i) = e^{\frac{s}{\tau_i}-1}, \quad s \in (-\infty, \tau_i].$$

Note that the distribution of scores pivots around the point where $s = 0$ and that $K_i(0|\tau_i) = e^{-1}$ regardless of which a_i^* is induced. If $\tau_i < \tau_j$ then an agent in group i therefore wins with probability

$$\int_{-\infty}^{\tau_i} K_j(s|\tau_i)^{n_j} K_i(s|\tau_i)^{n_i-1} dK_i(s|\tau_i),$$

which simplifies to $e^{n_j(\kappa_i-1)}\frac{1}{n_j\kappa_i+n_i}$. Similarly, if $\tau_i \geq \tau_j$ then an agent in group i therefore wins with probability

$$\int_{-\infty}^{\tau_j} K_j(s|\tau_i)^{n_j} K_i(s|\tau_i)^{n_i-1} dK_i(s|\tau_i) + \int_{\tau_j}^{\tau_i} K_i(s|\tau_i)^{n_i-1} dK_i(s|\tau_i),$$

which can be simplified to $\frac{1}{n_i}\left(1 - n_j e^{n_i\left(\frac{1}{\kappa_i}-1\right)}\frac{\kappa_i}{n_j\kappa_i+n_i}\right)$. Alternatively, the latter, where $\kappa_i \geq 1$, can be computed as follows. First, some member of group i wins only if no member of group j wins. The probability that no member of group j wins is $1 - n_j W(\kappa_j|n_j, n_i)$, where $\kappa_j = \frac{1}{\kappa_i} \in (0, 1)$. If a member of group j does not win, then any member of group i is equally likely to win, by group-symmetry. Hence, the probability that a given agent in group i wins is $\frac{1}{n_i}(1 - n_j W(\kappa_j|n_j, n_i))$.

It is straightforward to prove that $W(\kappa_i|n_i, n_j)$ is strictly increasing in κ_i and satisfies $W(1|n_i, n_j) = \frac{1}{n}$.

Finally, agents in group i are first claimants if $\kappa_i > 1$ or $\tau_i > \tau_j$. That is, they are guaranteed to outscore all agents in group j if their score exceeds the threshold $s_j(\overline{q}_j) = \tau_j$. The probability of this occurring is

$$\begin{aligned} 1 - K_i(\tau_j|\tau_i) &= 1 - e^{\frac{\tau_j}{\tau_i}-1} \\ &= 1 - e^{\frac{1}{\kappa_i}-1}, \end{aligned}$$

which is increasing in κ_i. ■

Proof of Corollary 2. Expected utility to a member of group i is

$$v_i W(\kappa_i | n_i, n_j) - c_i(a_i^*),$$

where a_i^* is a function of κ_i, as described implicitly in (12). In fact, from (12) it follows that

$$\frac{dc_i(a_i^*)}{d\kappa_i} = \frac{c_i'(a_i^*)}{t_i'(a_i^*)} v_i F'(\kappa_i | n_i, n_j).$$

Simple differentiation proves that $t_i'(a_i^*) \geq c_i'(a_i^*)$, given c_i is convex and f_i concave. Thus, $\frac{c_i'(a_i^*)}{t_i'(a_i^*)} \leq 1$, meaning that costs rise at a rate no higher than $v_i F'(\kappa_i | n_i, n_j)$. Thus, the rate of change in expected utility when κ_i changes is no smaller than

$$v_i \left(W'(\kappa_i | n_i, n_j) - F'(\kappa_i | n_i, n_j) \right).$$

This expression is independent of the technology. Indeed, it can be verified that it is positive when $n_i \geq 2$ (and when $n_i = 1$ and $\kappa_i > 1$, but not necessarily when $n_i = 1$ and $\kappa_i < 1$). The corollary follows. ■

Proof of Corollary 3. Given symmetric technologies,

$$\begin{aligned} s_1(q) - s_2(q) &= \tau_1 - \tau_2 + \tau_1 f(a_1) \ln H(q) - \tau_2 f(a_2) \ln H(q) \\ &= \tau_1 - \tau_2 + (\tau_1 f(a_1) - \tau_2 f(a_2)) \ln H(q) \\ &= \tau_2 \left[(\kappa_1 - 1) + (\kappa_1 f(a_1) - f(a_2)) \ln H(q) \right]. \end{aligned}$$

The slope of $s_1(q) - s_2(q)$ is determined by the sign of $\kappa_1 f(a_1) - f(a_2)$. Hence, $s_1(q) - s_2(q)$ is either weakly increasing or weakly decreasing and it thus crosses zero at most once. It therefore suffices to compare the corners, i.e. the extreme performances $\underline{q}$ and $\overline{q}$. First, $s_1(\overline{q}) - s_2(\overline{q}) = \tau_2 (\kappa_1 - 1)$ is positive if $\kappa_1 > 1$ and negative if $\kappa_1 < 1$. Second, $s_1(q) - s_2(q) \to \infty$ as $q \to \underline{q}$ if $\kappa_1 f(a_1) - f(a_2) < 0$ and $s_1(q) - s_2(q) \to -\infty$ if $\kappa_1 f(a_1) - f(a_2) > 0$.

When $\kappa_1 \geq 1$, it follows from Proposition 5 and $v_1 > v_2$ that $a_1 > a_2$. Hence, $\kappa_1 f(a_1) - f(a_2) > 0$. It follows that $s_1(q) - s_2(q)$ is first negative and then positive.

If $\kappa_1 < 1$ then $s_1(\overline{q}) - s_2(\overline{q})$ is negative. Therefore, if $\kappa_1 f(a_1) - f(a_2) > 0$ then $s_1(q) - s_2(q)$ is always negative. This conclusion applies if κ_1 is close to one. However, as κ_1 tends to zero, the term $\kappa_1 f(a_1) - f(a_2)$ tends to $-f(\overline{a}_2^s) < 0$.

Thus, when κ_1 is close to zero, $s_1(q) - s_2(q)$ is first positive and then negative. Since a_1 is strictly increasing in κ_1 and a_2 is strictly decreasing in κ_1, $\kappa_1 f(a_1) - f(a_2)$ is strictly increasing in κ_1. Thus, there is a unique value of κ_1 for which $\kappa_1 f(a_1) - f(a_2) = 0$. This is κ_1' in the statement of the corollary. ■

Proof of Proposition 7. At $\kappa_1 = \frac{v_1}{v_2} > 1$, it holds that $a_1^* > a_2^*$. Then, by Lemma 1, the MRT satisfies

$$MRT = -\frac{t'(a_1^*)}{t'(a_2^*)}\frac{n_1}{n_2} \leq -\frac{n_1}{n_2},$$

when $t(a_i)$ is convex. Convexity of t also implies that the feasible set is convex, again by Lemma 1. Given the objective is to maximize total effort, MRT≤MRS. The optimal action profile must therefore entail a weakly lower $a_1$and a weakly higher a_2 than what is implied by $\kappa_1 = \frac{v_1}{v_2}$. Thus, the optimal value of κ_1 must be no greater than $\frac{v_1}{v_2}$.

Next, $\kappa_1^s < 1$. If $\kappa_1^s > 0$ then at $\kappa_1 = \kappa_1^s$ it holds by definition that $a_1 = a_2$ and the MRT is therefore

$$MRT = -\frac{v_2}{v_1}\frac{n_1}{n_2}\kappa_1^s > -\frac{n_1}{n_2} = MRS$$

and it follows that the optimal value of κ_1 is strictly greater than κ_1^s. Hence, $a_1^* > a_2^*$ in equilibrium. If $\kappa_1^s = 0$, then MRT=0 and the same argument applies.

At $\kappa_1 = 1$, the MRT is

$$\begin{aligned} MRT &= -\frac{t'(a_1^*)}{t'(a_2^*)}\frac{v_2\frac{n-1}{n^2}}{v_1\frac{n-1}{n^2}}\frac{n_1}{n_2} \\ &= -\left(\frac{t'(a_1^*)}{t(a_1^*)}\bigg/\frac{t'(a_2^*)}{t(a_2^*)}\right)\frac{n_1}{n_2}, \end{aligned}$$

where $\frac{t'(a_i)}{t(a_i)}$ coincides with the derivative of $\ln t(a_i)$. If t is log-concave then $t'(a_i)$ is decreasing, and it follows that

$$MRT \geq -\frac{n_1}{n_2} = MRS,$$

which in turn implies that the optimal value of κ_1 is no smaller than 1.

Conversely, if t is locally log-convex, then there is an interval of actions such that if both a_1^* and a_2^* are in the interval and $a_1^* > a_2^*$, then $MRT < MRS$. Thus, if $\kappa_1 = 1$ induces actions in this interval, then the optimal value of κ_1 is below one. At $\kappa_1 = 1$, $t_i(a_i^*) = v_i \frac{n-1}{n^2}$, which means that a_i^* can be placed in the desired interval by increasing or decreasing v_i appropriately. This completes the proof. ■

Proof of Proposition 8. Convexity of t implies that the feasible set is convex, by Lemma 1. Monotonicity and quasi-concavity of $U_0(\mathbf{a})$ imply that the designer's indifference curve is convex. Thus, the optimal value of κ_1 is identified by the tangency condition.

Now, a key observation is that the MRS at $a_1 = a_2$ is $-\frac{n_1}{n_2}$, since the symmetry assumption implies that the marginal utility to the designer of any agent's action is the same at such a point. Thus, whenever the indifference curve meets the frontier of the feasible set at $a_1 > a_2$ – which requires $\kappa_1 > \kappa_1^s$ – the MRS is larger than $-\frac{n_1}{n_2}$ (closer to zero). Thus, at $\kappa_1 = \kappa_1^\Sigma > \kappa_1^s$, it holds by definition that $MRT = -\frac{n_1}{n_2}$ and therefore

$$MRT = -\frac{n_1}{n_2} \leq MRS.$$

Thus, $\kappa_1^U \leq \kappa_1^\Sigma$.

Next, as in Proposition 7, if $\kappa_1^s > 0$ then at $\kappa_1 = \kappa_1^s < 1$, $a_1 = a_2$ and

$$MRT = -\frac{v_2}{v_1}\frac{n_1}{n_2}\kappa_1^s > -\frac{n_1}{n_2} = MRS.$$

Thus, $\kappa_1^U > \kappa_1^s$. If $\kappa_1^s = 0$, then MRT$= 0$ and the same arguments apply.

The last statement of the proposition follows from the discussion prior to the proposition. In more detail, consider an indifference curve that meets the feasible set at $a_1 = a_2$, where $\kappa_1 = \kappa_1^s$. The former has slope $-\frac{n_1}{n_2}$ for all eligible $U_0(\mathbf{a})$. This action profile is feasible but not optimal. The more curvature the indifference curve has, the smaller is the superior set. If the superior set is small enough, then it only contains points on the frontier for which $\kappa_1 < 1$. In this case it follows trivially that $\kappa_1^U < 1$. Now, for a concrete example, assume that the designer has CES utility over $\mathbf{a}$. In the limit, this approaches Leontief for which the only point on the frontier of the feasible set that is in the superior set is the

$a_1 = a_2$ point that the indifference curve shares with the feasible set. Hence, with CES utility, there is always parameter values for which $\kappa_1^U < 1$. $\blacksquare$

Appendix B: Symmetric contests with a single costly prize

This appendix considers a special case of a Separable and Monotonic contest, as defined in Section 4. Specifically, it is assumed that there is a single prize, which is costly to the designer to award. The aim of the appendix is to explore how the optimal action profile and the optimal design depends on the cost. However, it is assumed for simplicity that all agents are identical and must be induced to take symmetric actions. Hence, subscripts are omitted.

Let $z \geq 0$ denote the cost of the prize to the designer. Although the feasible set is independent of z, the optimal action profile is now generally speaking both in the interior of the feasible set and sensitive to z. From (10), the prize is awarded only if the highest individual score exceeds z. Thus, there is a minimum standard q^z such that the prize is given out only if at least one agent performs above q^z. Note that q^z depends on the symmetric equilibrium action, a, and that (10) implies that $L(q^z|a) > 0$ when $a < \overline{a}^s$. It can be verified from the agents' first-order conditions that q^z is strictly decreasing in a for all $a < \overline{a}^s$.[20] Stated differently, a lower standard and a higher equilibrium action go hand in hand. This in turn means that the probability that the prize is awarded, $1 - G(q^z|a)^n$, increases when a higher action is induced.

The designer's problem is to induce an action a to maximize expected payoff

$$U_0(a, a, ..., a) - z(1 - G(q^z|a)^n).$$

Now consider an increase in cost z. This makes it less attractive to award the prize. Given the conclusion in the previous paragraph, this implies that the designer will induce a lower action. In summary, when the prize is costlier to the designer, she induces a lower action by imposing a higher standard. However, when the cost of the prize becomes too high, the designer is better off shutting down the contest.

[20] When $a < \overline{a}^s$, the action profile is in the interior of the feasible set. In contrast, Proposition 2 considers action profiles along the frontier of the feasible set and a minimum standard that is found where the LR is zero.

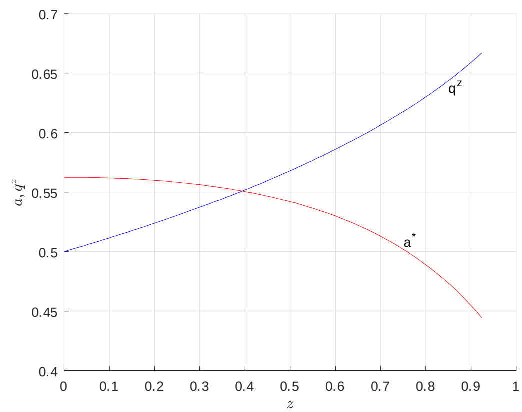

Figure 3: The equilibrium standard and equilibrium action as a function of costs.

As an example, consider a special case of the spanning model, with

$$G(q|a) = \sqrt{a}q^2 + \left(1 - \sqrt{a}\right) q, \ q \in [0, 1],$$

whenever $a \in [0, 1)$. Assume $v = 6$ and $c(a) = a$. If the prize is costless, or $z = 0$, the optimal minimum standard is $\widehat{q} = \frac{1}{2}$. The highest action that can be implemented is then $\overline{a} = \frac{9}{16}$.

For simplicity, assume that there is exactly one contestant or agent. Thus, the agent wins the prize if and only if his performance exceeds the minimum standard. For any given minimum standard q^z, the first-order condition implies that the agent's best response is $a = 9 \left(q^z\right)^2 (1-q^z)^2$, which is of course maximized if $q^z = \widehat{q} = \frac{1}{2}$. However, it may be better to increase q^z in order to lower the probability that the designer has to incur the cost of awarding the prize. Whenever $q^z > \frac{1}{2}$, a and q^z move in opposite directions.

The probability that the prize is awarded is $1 - G(q^z|a) = 3 \left(q^z\right)^4 - 6 \left(q^z\right)^3 + 3 \left(q^z\right)^2 - (q^z) + 1$. Assume that the designer wishes to maximize the agent's action, or $U_0(a) = a$. Then, the designer's expected payoff is

$$9 \left(q^z\right)^2 (1 - q^z)^2 - z \left(3 \left(q^z\right)^4 - 6 \left(q^z\right)^3 + 3 \left(q^z\right)^2 - (q^z) + 1\right).$$

At $z = \frac{12}{13}$, this is maximized at $q^z = \frac{2}{3}$ and at exactly zero expected payoff.

Thus, it is optimal for the designer to shut down the contest if $z > \frac{12}{13}$, which can be achieved by imposing a minimum standard of $q = 1$. On the other hand, as long as $z \in \left(0, \frac{12}{13}\right)$, a minimum standard in the interval $\left(\frac{1}{2}, \frac{2}{3}\right)$ is optimal and this minimum standard is increasing in z. Figure 3 illustrates the solution.

Note that $z = \frac{12}{13}$ is substantially higher than the action that is induced at $q^z = \frac{2}{3}$, which is $a = \frac{4}{9}$. Even though the price is extremely expensive to the designer, the contest is still profitable when z is just below $\frac{12}{13}$ because there is only a small chance that the prize must be awarded.

Appendix C: The best-shot model

This appendix complements the treatment of the best-shot model in Section 5. Among other things, the CSF is computed and the feasible set is constructed when rationing is possible. It concludes with a discussion of how to microfound the biased lottery CSF that is often used in the current literature, and whether this microfoundation is appealing or desirable.

C.1 Winning probabilities and CSFs

As mentioned in Section 6.2, it is possible to extend the model to more than two groups of agents. The main complication is that there are now many types of "sequential" allocation rules of the kind described at the beginning of Section 3.3. For example, group 1 might be "served" first, followed then by group 2 and later by group 3, while all remaining groups at the very end fight each other simultaneously if the prize is still available. A complete description of the feasible set requires one to piece together all these cases. See Kirkegaard (2020) for details.

This subsection instead considers the simplest possible case in which the allocation rule is "simultaneous". Thus, every agent receives a score of the form $\mu_i v_i L_i(q_i|a_i)$, with $\mu_i \in (0,\infty)$ for all $i \in N$, and the agent with the highest score wins. Using the notation from Section 5.1, this means that $\tau_i \in (0,\infty)$. It turns out that τ_i is in a sense a measure of how favorable the contest is to agent i, as demonstrated in the following result.

Proposition 9 *Consider the best-shot model with an arbitrary number of agents. Fix an action profile* $\mathbf{a}^*$ *on the frontier of the feasible set in which all agents are active (*$a_i > 0$*) and which is implemented by giving each agent a scoring function* $s_i(q_i) = \mu_i v_i L_i(q_i|a_i^*)$*, with* $\mu_i \in (0,\infty)$ *for all* $i \in N$*. Then, agent* i*'s ex ante equilibrium winning probability exceeds that of agent* j *if and only if* $\tau_i > \tau_j$*, regardless of whether rationing is allowed or not.*

Proof. Note that if agents i and j perform equally well given what is expected of them – i.e. they perform at the same quantiles, or $G_i(q_i|a_i^*) = G_j(q_j|a_j^*)$ – then agent i's score beats agent j's score if $\tau_i > \tau_j$ and the likelihood-ratios are positive. However, agent's j's score is higher if the likelihood-ratios are negative.

Consequently, the result is trivial if rationing is allowed. Then, only positive likelihood-ratios have a chance of winning. Recall that agents i and j have positive likelihood-ratios with the same probability, specifically $1-e^{-1}$. Given a performance at any fixed quantile above e^{-1}, such that $G_i(q_i|a_i^*) = G_j(q_j|a_j^*) \geq e^{-1}$, agent i outscores agent j if and only if $\tau_i > \tau_j$. Since quantiles are distributed the same way (uniformly) for both agents, it now follows that agent i wins with a higher probability in equilibrium if and only if $\tau_i > \tau_j$.

If rationing is ruled out, then performance with negative likelihood-ratio come into play. Given τ_i, agent i's score is in equilibrium distributed according to

$$K_i(s_i|\tau_i) = e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i]$$

with density

$$k_i(s_i|\tau_i) = \frac{1}{\tau_i}e^{\frac{s_i}{\tau_i}-1},\ s_i \in (-\infty, \tau_i].$$

Without loss of generality, arrange agents in ascending order based on their τ_i, with $\tau_1 \leq \tau_2 \leq ...\tau_N$. Let $\tau_0 = -\infty$. A score above τ_j automatically beats agent j. Hence, agent i's equilibrium winning probability can then be written as

$$\begin{aligned}
P_i^*(\tau_1, \tau_2, ..., \tau_N) &= \int_{\tau_0}^{\tau_1} \left(\prod\nolimits_{j\geq 1, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i)ds \\
&\quad + \int_{\tau_1}^{\tau_2} \left(\prod\nolimits_{j\geq 2, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i)ds \\
&\quad +... + \int_{\tau_{i-1}}^{\tau_i} \left(\prod\nolimits_{j\geq i, j\neq i} K_j(s|\tau_j)\right) k_i(s|\tau_i)ds \\
&= \frac{1}{\tau_i}\sum_{m=1}^{i} \alpha_m,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_m &= \int_{\tau_{m-1}}^{\tau_m} e^{\sum_{j\geq m}\left(\frac{s}{\tau_j}-1\right)} ds \\
&= \frac{1}{\sum_{j\geq m}\frac{1}{\tau_j}} \left(e^{\sum_{j\geq m}\left(\frac{\tau_m}{\tau_j}-1\right)} - e^{\sum_{j\geq m}\left(\frac{\tau_{m-1}}{\tau_j}-1\right)}\right).
\end{aligned}$$

Going forward, for $i = 2, ..., n$, it is useful to compare

$$\alpha_i = \frac{1}{\sum_{j \geq i} \frac{1}{\tau_j}} \left(e^{\sum_{j \geq i} \left(\frac{\tau_i}{\tau_j} - 1 \right)} - e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)} \right)$$

and

$$\begin{aligned} \sum_{m=1}^{i-1} \alpha_m &\leq \int_{\tau_0}^{\tau_{i-1}} e^{\sum_{j \geq i-1} \left(\frac{s}{\tau_j} - 1 \right)} ds. \\ &= \frac{1}{\sum_{j \geq i-1} \frac{1}{\tau_j}} e^{\sum_{j \geq i-1} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)} \\ &= \frac{1}{\sum_{j \geq i-1} \frac{1}{\tau_j}} e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)}. \end{aligned}$$

Then, note that for $i = 2, ..., n$,

$$\begin{aligned} P_i^* - P_{i-1}^* &= \frac{1}{\tau_i} \sum_{m=1}^{i} \alpha_m - \frac{1}{\tau_{i-1}} \sum_{m=1}^{i-1} \alpha_m \\ &= \frac{1}{\tau_i} \alpha_i - \left(\frac{1}{\tau_{i-1}} - \frac{1}{\tau_i} \right) \sum_{m=1}^{i-1} \alpha_m \\ &\geq \frac{1}{\tau_i} \frac{1}{\sum_{j \geq i} \frac{1}{\tau_j}} \left(e^{\sum_{j \geq i} \left(\frac{\tau_i}{\tau_j} - 1 \right)} - e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)} \right) \\ &\quad - \left(\frac{1}{\tau_{i-1}} - \frac{1}{\tau_i} \right) \frac{1}{\sum_{j \geq i-1} \frac{1}{\tau_j}} e^{\sum_{j \geq i} \left(\frac{\tau_{i-1}}{\tau_j} - 1 \right)}, \end{aligned}$$

and where, defining $x_i = \sum_{j \geq i} \frac{1}{\tau_j}$, the latter is proportional to

$$\begin{aligned} \Delta_i &= (1 + \tau_{i-1} x_i) \left(e^{\tau_i x_i} - e^{\tau_{i-1} x_i} \right) - (\tau_i - \tau_{i-1}) x_i e^{\tau_{i-1} x_i} \\ &= (1 + \tau_{i-1} x_i) e^{\tau_i x_i} - (1 + \tau_i x_i) e^{\tau_{i-1} x_i} > 0 \end{aligned}$$

when $\tau_i > \tau_{i-1}$. Hence, it now follows that winning probabilities are arranged in the same order as the τ_i's. ■

Given a vector $\boldsymbol{\tau}$ that lists all τ_i's, it is in principle possible to derive the CSF – the probability that agent i wins for any given action profile $\mathbf{a}$ – by integrating out the uncertainty over performance, i.e. by calculating

$$\int \left(\int P_i(q_i, \mathbf{q}_{-i}) g_i(q_i|a_i) dq_i \right) \prod_{j \neq i} g_j(q_j|a_j) d\mathbf{q}_{-i}.$$

In the best-shot model, however, a more direct argument is also possible. This is illustrated in the proof of the next proposition, under the assumption that rationing is ruled out and that all agents are active. In this case, negative scores have a chance of winning.

Proposition 10 *Under the assumptions in Proposition 9, if* $\mathbf{a}^*$ *is the equilibrium action profile and* $\tau_i \leq \tau_j$ *for all* $j \in N$*, then agent* i *wins with probability*

$$\widehat{p}_i(\mathbf{a}|\boldsymbol{\tau}) = \left(\prod\nolimits_{j \in N \setminus \{i\}} e^{\frac{(\tau_i - \tau_j) f_j(a_j)}{\tau_j f_j(a_j^*)}} \right) \frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j \in N} \frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \tag{22}$$

when rationing is ruled out, for any action profile $\mathbf{a}$ *with* $a_i > 0$*.*

Proof. To start, note that the distribution of agent i's score is

$$S_i(s|a_i) = \left(e^{s-\tau_i}\right)^{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}, \quad s \in (-\infty, \tau_i]$$

when he takes action a_i rather than a_i^*. It is as if he draws $\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}$ scores from the distribution $e^{s-\tau_i}$, but only the best score is counted. The range of scores depends on the identity of the agent, with τ_i describing the highest possible score that agent i can achieve. Assume agent i is the agent with the lowest τ, or $\tau_i \leq \tau_j$. Then, in order for agent i to win it is necessary that all other agents score below τ_i, the probability of which is

$$\left(\prod\nolimits_{j \in N \setminus \{i\}} e^{\frac{(\tau_i - \tau_j) f_j(a_j)}{\tau_j f_j(a_j^*)}} \right). \tag{23}$$

Given this event, however, the conditional distribution of agent j's score is

$$\frac{S_j(s|a_j)}{S_j(\tau_i|a_j)} = \left(e^{s-\tau_i}\right)^{\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}, \quad s \in (-\infty, \tau_i].$$

Hence, it is as if all agents draw scores from the *same* distribution, $e^{s-\tau_i}$. Since each draw therefore has an equal chance of winning, the conditional probability that agent i wins is

$$\frac{\frac{f_i(a_i)}{\tau_i f_i(a_i^*)}}{\sum_{j\in N}\frac{f_j(a_j)}{\tau_j f_j(a_j^*)}}. \tag{24}$$

Combining (23) and (23) yields the CSF in the proposition. ■

As a consistency check, note that if $\tau_i = \tau_j$ for all $j \in N$ then $\widehat{p}_i(\mathbf{a}^*|\boldsymbol{\tau}) = \frac{1}{n}$ and all agents win with equal probability in equilibrium. Note that the first term in (22) depends on the action profile, for reasons that are carefully explained in the proof of the proposition. Due to this distortion, (22) is *not* a lottery CSF (except in the special case where $\tau_i = \tau_j$ for all $j \in N$).

C.2 The feasible set with and without rationing

Next, the feasible set of implementable actions is characterized. As explained in the main text, it is assumed that

$$\lim_{a_i\to 0} c_i'(a_i)\frac{f_i(a_i)}{f_i'(a_i)} = 0,$$

for all i. Now, the highest possible implementable action of agent i, $\overline{a}_i$, can be characterized succinctly in the best-shot model. This follows from the proof of Proposition 1.

Corollary 4 *In the best-shot model, any action no greater than the unique solution $\overline{a}_i$ to*

$$c_i'(\overline{a}_i)\frac{f_i(\overline{a}_i)}{f_i'(\overline{a}_i)} = \frac{v_i}{e} \tag{25}$$

can be implemented by appropriately designing the assignment rule.

Proof. In the best-shot model, where $\widehat{q}_i(a_i^t) = H^{-1}\left(e^{-\frac{1}{f_i(a_i^t)}}\right)$ or $H(\widehat{q}_i(a_i)) = e^{-\frac{1}{f_i(a_i^t)}}$, (17) is

$$\overline{U}_i(a_i) = v_i\left(1 - e^{-\frac{f_i(a_i)}{f_i(a_i^t)}}\right) - c_i(a_i)$$

and (18) simplifies to

$$\frac{1}{c_i'(a_i^t)}\frac{f_i'(a_i^t)}{f_i(a_i^t)} \geq \frac{e}{v_i}.$$

By concavity of f_i and convexity of c_i, the left hand side is decreasing. Hence, the condition is satisfied if and only a_i^t is no greater than the solution to (25). By Proposition 1, it is then possible to implement the action. ■

Using similar logic, it is possible to characterize the corners of the frontier of the feasible set when there are two groups of agents and the rules are group-symmetric. As in Section 3, let $\overline{a}_i^s$ denote the highest possible action is group i when rules are group-symmetric. When $n_i = 1$, $\overline{a}_i^s = \overline{a}_i$ but otherwise $\overline{a}_i^s < \overline{a}_i$. Similarly, let $\underline{a}_i^s$ denote the smallest possible action along the frontier for an agent in group i when rules are group-symmetric. This is the action that is implemented when $a_j = \overline{a}_j^s$ in the other group, $j \neq i$. This means that an agent in group i has a chance of winning only if all agents in group j have negative likelihood-ratios. If $n_i \geq 2$, then competition within group i still ensures that $\underline{a}_i^s > 0$. However, if $n_i = 1$, then agent i is simply the "residual claimant" of the prize and has no incentive to exert effort.

Corollary 5 *In the best-shot model with two groups, group-symmetric rules, and no rationing, the frontier of the feasible set contains the corners $(\overline{a}_1^s, \underline{a}_2^s)$ and $(\underline{a}_1^s, \overline{a}_2^s)$, where $\overline{a}_i^s$ and $\underline{a}_i^s$ solve*

$$c_i'(\overline{a}_i^s)\frac{f_i(\overline{a}_i^s)}{f_i'(\overline{a}_i^s)} = v_i\frac{n_i - 1 + e^{-n_i}}{n_i^2} \text{ and } c_i'(\underline{a}_i^s)\frac{f_i(\underline{a}_i^s)}{f_i'(\underline{a}_i^s)} = v_i e^{-n_j}\frac{n_i - 1}{n_i^2}, \quad i, j = 1, 2 \text{ and } j \neq i.$$

Here, $\overline{a}_i^s$ is strictly decreasing in n_i and independent of n_j. Similarly, $\underline{a}_i^s$ is strictly positive if and only if $n_i \geq 2$ and it is then strictly decreasing in both n_i and n_j.

Proof. To implement $\overline{a}_i^s$, the contest rules must imply that an agent in group i wins if and only if his likelihood-ratio is positive and (by group-symmetry and the MLRP) if his performance is higher than the performance of all other agents in his group. Given all other agents in group i takes action $\overline{a}_i^s$, the relevant first-order condition is

$$\begin{aligned} c_i'(\overline{a}_i^s) &= v_i \int_{\widehat{q}_i(\overline{a}_i^s)}^{\overline{q}_i} (G_i(q_i|\overline{a}_i^s))^{n_i-1} L_i(q_i|\overline{a}_i^s) g_i(q_i|\overline{a}_i^s) dq_i \\ &= v_i \frac{f_i'(\overline{a}_i^s)}{f_i(\overline{a}_i^s)} \int_{\widehat{q}_i(\overline{a}_i^s)}^{\overline{q}_i} (G_i(q_i|\overline{a}_i^s))^{n_i-1} (1 + \ln G_i(q_i|\overline{a}_i^s)) g_i(q_i|\overline{a}_i^s) dq_i \end{aligned}$$

Substituting the equilibrium quantiles of agent i's performance, $z = G_i(q_i|\overline{a}_i^s)$ and $dz = g_i(q_i|\overline{a}_i^s)dq_i$, yields

$$\begin{aligned} c_i'(\overline{a}_i^s) \frac{f_i(\overline{a}_i^s)}{f_i'(\overline{a}_i^s)} &= v_i \int_{e^{-1}}^{1} z^{n_i-1} (1 + \ln z) dz \\ &= v_i \frac{n_i - 1 + e^{-n_i}}{n_i^2}. \end{aligned}$$

Simple differentiation shows that the right hand side is decreasing in n_i.

Turning to $\underline{a}_i^s$, this is the action that is implemented when the other group is induced to take action $a_j = \overline{a}_j^s$, $j \neq i$. Thus, an agent in group i has a chance of winning only if all agents in group j have negative likelihood-ratios. This occurs with probability e^{-n_j}. Conditional on this event, the agent must moreover (by group-symmetry and the MLRP) outperform all other agents in his group. Since rationing is ruled out, such an agent may win even if his own likelihood-ratio is negative. Thus, the first-order condition is

$$c_i'(\underline{a}_i^s) = v_i e^{-n_j} \int_{\underline{q}_i}^{\overline{q}_i} (G_i(q_i|\underline{a}_i^s))^{n_i-1} L_i(q_i|\underline{a}_i^s) g_i(q_i|\underline{a}_i^s) dq_i,$$

which reduces to the statement in the corollary. The rest follows by simple differentiation. ■

Next, the interior part of the frontier is characterized.

Proposition 11 *In the best-shot model with two groups, group-symmetric rules, and no rationing, the frontier of the feasible set contains the corners* $(\overline{a}_1^s, \underline{a}_2^s)$ *and* $(\underline{a}_1^s, \overline{a}_2^s)$. *The remaining action profiles on the frontier can be traced out by varying* $\tau_1 > 0$ *and* $\tau_2 > 0$, *where the equilibrium action of an agent in group* i

is determined by

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right),$$

with

$$F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) = \begin{cases} e^{n_j\left(\frac{\tau_i}{\tau_j}-1\right)} \frac{n_j\frac{\tau_i}{\tau_j}+n_i-1}{\left(n_j\frac{\tau_i}{\tau_j}+n_i\right)^2} & \text{if } \frac{\tau_i}{\tau_j} \in (0,1) \\ \frac{n_i-1}{n_i^2} + e^{n_i\left(\frac{1}{\tau_i/\tau_j}-1\right)} \frac{n_j}{n_i^2} \frac{n_j\left(\frac{\tau_i}{\tau_j}\right)^2 + \frac{\tau_i}{\tau_j} n_i(2-n_j) - n_i^2}{\left(n_j\frac{\tau_i}{\tau_j}+n_i\right)^2} & \text{if } \frac{\tau_i}{\tau_j} \geq 1 \end{cases}.$$

Here, $F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right)$ *is strictly increasing in* $\frac{\tau_i}{\tau_j}$ *and satisfies* $F(1|n_i, n_j) = \frac{n-1}{n^2}$*. Hence,* a_i^* *is strictly increasing in* $\frac{\tau_i}{\tau_j}$*.*

Proof. The corners are described in Corollary 5. To describe actions away from the corners of the frontier, note that such actions must be interior and the two first-order conditions must therefore be solved simultaneously. Assume first that $0 < \tau_i \leq \tau_j$. Then, regardless of his performance, an agent in group i wins with a probability strictly less than one when $\tau_i < \tau_j$. If his performance is q_i, then he beats an agent in group j if and only if $s_i(q_i) \geq s_j(q_j)$, which occurs if and only if q_i and q_j are such that

$$e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}} \geq G_j(q_j|a_j^*),$$

where the term on the right hand side is the equilibrium distribution of the performance of a member of group j. Hence, the interim probability that agent i with performance q_i beats such an agent is $e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}$. To win, the agent has to beat all agents in group j as well as all the other agents in group i. With this in mind, agent i's first-order condition in equilibrium is

$$v_i \int_{\underline{q}_i}^{\overline{q}_i} L_i(q_i|a_i^*) \left(e^{\frac{\tau_i-\tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}\right)^{n_j} G_i(q_i|a_i^*)^{n_i-1} g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0$$

or

$$v_i \frac{f_i'(a_i^*)}{f_i(a_i^*)} e^{n_j \frac{\tau_i - \tau_j}{\tau_j}} \int_{\underline{q}_i}^{\overline{q}_i} \left(1 + \ln G_i(q_i|a_i^*)\right) G_i(q_i|a_i^*)^{n_j \frac{\tau_i}{\tau_j} + n_i - 1} g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0.$$

As in Corollary 5, substituting the equilibrium quantiles of agent i's performance, $z = G_i(q_i|a_i^*)$ and $dz = g_i(q_i|a_i^*)dq_i$. This gives

$$\begin{aligned} c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} &= v_i e^{n_j \frac{\tau_i - \tau_j}{\tau_j}} \int_0^1 \left(1 + \ln z\right) z^{n_j \frac{\tau_i}{\tau_j} + n_i - 1} dz \\ &= v_i e^{n_j \left(\frac{\tau_i}{\tau_j} - 1\right)} \frac{n_j \frac{\tau_i}{\tau_j} + n_i - 1}{\left(n_j \frac{\tau_i}{\tau_j} + n_i\right)^2}, \end{aligned}$$

which nails down a_i^* since the left hand side is strictly increasing in a_i^*.

Assume now that $\tau_i > \tau_j > 0$. In this case, agent i beats any agent in group j with probability one if his performance is high enough, or specifically if $q_i \geq \widetilde{q}_i$ where

$$e^{\frac{\tau_i - \tau_j}{\tau_j}} G_i(\widetilde{q}_i|a_i^*)^{\frac{\tau_i}{\tau_j}} = 1,$$

which implies that

$$G_i(\widetilde{q}_i|a_i^*) = e^{\frac{\tau_j - \tau_i}{\tau_i}}$$

and

$$1 + \ln G_i(\widetilde{q}_i|a_i^*) = \frac{\tau_j}{\tau_i}.$$

Agent i's first order condition is now

$$\begin{aligned} &v_i \int_{\underline{q}_i}^{\widetilde{q}_i} L_i(q_i|a_i^*) \left(e^{\frac{\tau_i - \tau_j}{\tau_j}} G_i(q_i|a_i^*)^{\frac{\tau_i}{\tau_j}}\right)^{n_j} G_i(q_i|a_i^*)^{n_i - 1} g_i(q_i|a_i^*) dq_i \\ &\qquad + v_i \int_{\widetilde{q}_i}^{\overline{q}_i} L_i(q_i|a_i^*) G_i(q_i|a_i^*)^{n_i - 1} g_i(q_i|a_i^*) dq_i - c_i'(a_i^*) = 0. \end{aligned}$$

The same substitution as before yields

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i e^{n_j \frac{\tau_i - \tau_j}{\tau_j}} \int_0^{e^{\frac{\tau_j - \tau_i}{\tau_i}}} \left(1 + \ln z\right) z^{n_j \frac{\tau_i}{\tau_j} + n_i - 1} dz + v_i \int_{e^{\frac{\tau_j - \tau_i}{\tau_i}}}^1 \left(1 + \ln z\right) z^{n_i - 1} dz$$

or

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i \left(\frac{n_i - 1}{n_i^2} + e^{n_i\left(\frac{1}{\tau_i/\tau_j}-1\right)} \frac{n_j}{n_i^2} \frac{n_j \left(\frac{\tau_i}{\tau_j}\right)^2 + \frac{\tau_i}{\tau_j} n_i \left(2 - n_j\right) - n_i^2}{\left(n_j \frac{\tau_i}{\tau_j} + n_i\right)^2} \right).$$

As before, a_i^* is nailed down because the left hand side is strictly increasing. The characterization result in the proposition now follows. The fact that $F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right)$ is strictly increasing in $\frac{\tau_i}{\tau_j}$ is verified by differentiation, and it is straightforward to verify that

$$F(1|n_i, n_j) = \frac{n-1}{n^2}.$$

■

It can be verified that as $\frac{\tau_i}{\tau_j}$ converges to infinity or zero, a_i converges to $\overline{a}_i^s$ and $\underline{a}_i^s$ as described in Corollary 5, respectively.

The frontier of the feasible set is described in a similar fashion when rationing is allowed.

Proposition 12 *In the best-shot model with two groups, group-symmetric rules, and with rationing allowed, the action profiles on the frontier of the feasible set, away from the corners, can be traced out by varying $\tau_1 > 0$ and $\tau_2 > 0$. The equilibrium action of an agent in group i is determined by*

$$c_i'(a_i^*)\frac{f_i(a_i^*)}{f_i'(a_i^*)} = v_i F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right),$$

where

$$F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) = F\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right) + \frac{e^{-(n_i+n_j)}}{\left(n_j\frac{\tau_i}{\tau_j} + n_i\right)^2}.$$

Here, $F_R\left(\frac{\tau_i}{\tau_j}|n_i, n_j\right)$ is strictly increasing in $\frac{\tau_i}{\tau_j}$ and satisfies $F_R\left(1|n_i, n_j\right) = \frac{n-1}{n^2} + \frac{e^{-n}}{n^2}$. Hence, a_i^* is strictly increasing in $\frac{\tau_i}{\tau_j}$.

Proof. The proof follows the same steps as in the proof of Proposition 11. The only difference is that an agent in group i now has zero probability of winning if $q_i < \widehat{q}_i(a_i^*)$ or, using the same substitution as in Proposition 11, if

$z \leq G_i(\widehat{q}_i(a_i^*)|a_i^*) = e^{-1}$. Hence, the lower bounds on the integrals that are evaluated in the proof of Proposition 11 change. This produces $F_R(\frac{\tau_i}{\tau_j})$ as stated in the proposition. Monotonicity can be verified by differentiation. ■

Note that $F_R(\frac{\tau_i}{\tau_j}) > F(\frac{\tau_i}{\tau_j})$. Since t_i is increasing, it follows, as expected, that the action profile for any given $\frac{\tau_i}{\tau_j}$ is higher when rationing is allowed than when it is not. It can be verified that Lemma 1 is unchanged when rationing is allowed.

Winning probabilities can be computed using the method described in the proof of Proposition 6, except the integration is performed only over non-negative scores. With $\kappa_i = \frac{\tau_i}{\tau_j}$, this yields equilibrium winning probabilities for an agent in group i of

$$W_R\left(\kappa_i|n_i,n_j\right) = \begin{cases} e^{n_j(\kappa_i-1)}\frac{1}{n_j\kappa_i+n_i} - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i+n_i} & \text{if } \kappa_i \in (0,1) \\ \frac{1}{n_i}\left(1-n_j e^{n_i\left(\frac{1}{\kappa_i}-1\right)}\frac{\kappa_i}{n_j\kappa_i+n_i}\right) - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i+n_i} & \text{if } \kappa_i \geq 1 \end{cases}$$

or simply

$$W_R\left(\kappa_i|n_i,n_j\right) = W\left(\kappa_i|n_i,n_j\right) - \frac{e^{-(n_i+n_j)}}{n_j\kappa_i+n_i}, \ \kappa_i \in (0,\infty).$$

With the proof of Corollary 2 in mind, it can be verified that

$$W_R\left(\kappa_i|n_i,n_j\right) - F_R\left(\kappa_i|n_i,n_j\right)$$

is increasing in κ_i whenever $W\left(\kappa_i|n_i,n_j\right) - F\left(\kappa_i|n_i,n_j\right)$ is. The only case in which $W\left(\kappa_i|n_i,n_j\right) - F\left(\kappa_i|n_i,n_j\right)$ is not increasing is when $n_i = 1$ and $\kappa_i < 1$. Checking this case, however, it turns out that $W_R\left(\kappa_i|n_i,n_j\right) - F_R\left(\kappa_i|n_i,n_j\right)$ is in fact increasing. In other words, $W_R\left(\kappa_i|n_i,n_j\right) - F_R\left(\kappa_i|n_i,n_j\right)$ is globally increasing in κ_i for all (n_i,n_j). By the argument in Corollary 2, it follows that expected utility to agents in group i is strictly increasing in κ_i.

C.3 Microfoundations for the biased lottery CSF

It is possible to use the best-shot model to provide microfoundations for (15).

Proposition 13 *Consider the best-shot model with $H_i(q_i) = H(q_i)$, $i \in N$. As-*

sign agent i with performance q_i a base score of $s_i^B(q_i) = H(q_i)^{1/b_i} \in [0,1]$, $b_i > 0$. Draw an auxiliary score s_i^{AUX} for agent i from the distribution $\left(s_i^{AUX}\right)^{\delta_i}$, $s_i^{AUX} \in [0,1]$, $\delta_i \geq 0$. Let agent i's final score be $s_i^{FM}(q_i) = \max\{s_i^B(q_i), s_i^{AUX}\}$. Finally, draw a score s^D for the designer from the distribution $\left(s^D\right)^z$, $s^D \in [0,1]$, $z \geq 0$. Let the individual (agent or designer) with the highest score win. Then, the CSF is given by (15).

Proof. Agent i's final score is below s_i if and only if both s_i^B and s_i^{AUX} are below s_i. First, $s_i^B \leq s_i$ when $q_i \leq H^{-1}\left(s_i^{b_i}\right)$, the probability of which is $H(q_i)^{f_i(a_i)} = s_i^{b_i f_i(a_i)}$. Second, the probability that $s_i^{AUX} \leq s_i$ is $s_i^{\delta_i}$. Hence, the probability that the final score is below s_i is $s_i^{b_i f_i(a_i)} s_i^{\delta_i} = s_i^{b_i f_i(a_i)+\delta_i}$. It is as if agent i draws $b_i f_i(a_i) + \delta_i$ "ideas" from a uniform distribution. Similarly, the designer draws z "ideas" from a uniform distribution. Since each "idea" is equally likely to be the best, the ex ante probability that agent i wins is (15). ∎

The transformation of q_i into a base score maps the idea from the support $[\underline{q}, \overline{q}]$ into a quality index on $[0,1]$, where the index is identity dependent via b_i. Given action a_i, agent i then draws $b_i f_i(a_i)$ ideas from a uniform distribution on this index. He is then given δ_i fake ideas by the designer, again drawn from a uniform distribution. The agent now has a total of $b_i f_i(a_i) + \delta_i$ real and fake ideas. The designer also draws z fake ideas from a uniform distribution. Each idea, real or fake, has an equal chance of winning, yielding (15).

The stochastic nature of the fake ideas may or may not be palatable. Thus, Proposition 13 should not be taken as a defense of (15) but rather as a clarification of the lengths one must go to in order to justify it. The transformation of the performance into a quality index seems more appealing. However, this particular transformation is still ad hoc.[21] In fact, Proposition 13 merely shows that (15) can be implemented in the Fullerton and McAfee (1999) model. Hence, it follows that the set of implementable actions must in reality be strictly larger than the set of actions that can be implemented by using (15).[22]

[21] Similarly, giving agents a multiplicative bonus in Hirschleifer and Riley's (1992) model yields $p_i(\mathbf{a}|\mathbf{0}, \mathbf{b}, 0)$. This can also be obtained by variying the β_i parameter in Clark and Riis' (1996) random utility framework. Again, these are ad hoc ways to manipulate the contest.

[22] Fu and Wu (2020) and most of the prior literature restrict head starts to be non-negative. Drugov and Ryvkin (2017) show that negative head starts may be better. However, negative head starts cannot be justified by Proposition 13.

One drawback of using (15) for contest design is that it says little about how to implement the optimal design in practice. For instance, how exactly is the playing field supposed to be made level if the designer does not observe actions? Proposition 13 tells us how this can be achieved by linking design to the observable signals. In other words, the kind of story embodied in Proposition 13 is important if the desire is to apply lessons from (15) in practice. The issue is that (15) pushes the performance profile to the back, which is unfortunate since this is the observable variable. The stochastic performance approach in the current paper has the distinct advantage that it starts directly from the observables.