Efficiency in Asymmetric Auctions with Endogenous Reserve Prices*

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Abstract

In the standard independent private values model, the second-price auction (SPA) is generally taken to be more efficient than the first-price auction (FPA) when bidders are asymmetric. However, this conclusion assumes that reserve prices are identical across auctions. This paper endogenizes the reserve price and shows that it may be lower in the FPA. Hence, gains from trade are realized more often in the FPA. This effect may make the FPA more efficient than the SPA. Indeed, the FPA may Pareto dominate the SPA. That is, the FPA may be more profitable and yet be preferred by all bidders.

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1 Introduction

A central tenet of auction theory is that all commonly used auctions are equally profitable and equally efficient when bidders are symmetric, risk neutral, and have independent private values. This paper relaxes the symmetry assumption. Even with asymmetric bidders, the second-price auction (SPA) allocates the object efficiently whenever it is sold. Thus, the SPA is *conditionally efficient*: Conditional on a sale, gains from trade are maximized. Since the first-price auction (FPA) does not have this property it is tempting to conclude that the SPA is more efficient than the FPA. Indeed, most of the theoretical and empirical literature studies the apparently more intricate question of which auction is more profitable. Nevertheless, the efficiency question deserves more attention. Not only is efficiency and distributional concerns relevant to society and governments alike, they also inform regulation as well.

As Hu, Matthews, and Zou (2010) among many others point out, in any given auction the possibility that the object may not be traded due for example to a reserve price leads to efficiency loss on its own. It is for this reason that the efficiency question cannot immediately be put to rest. After all, focusing only on conditional efficiency ignores the possibility that the object may not be sold with the same probability in the two auctions. In either auction, the object is sold if and only if there is at least one bidder whose valuation exceeds the reserve price. Hence, if the reserve price is different in the SPA and the FPA then gains from trade are not realized equally often.

This paper considers a seller who designs the reserve price to maximize expected revenue without regard to efficiency. The resulting reserve price may be lower in the FPA than in the SPA, in which case the FPA generates gains from trade more often. Hence, it is no longer obvious which auction is more efficient. If the reserve price is much lower in the FPA than in the SPA, the former may be more efficient.

In fact, it turns out that the FPA may even Pareto dominate the SPA ex ante. That is, the seller and all the bidders, be they strong or weak, may agree that the FPA is preferred. This should be seen in light of the common assertion, originally due to the seminal paper by Maskin and Riley (2000), that the SPA is preferred by strong bidders. The reason is that the FPA tends to favor the weak bidders to the detriment of the strong bidders. However, when the reserve price is lowered it benefits types that would otherwise have been excluded, even if those types belong to strong bidders. If the difference between reserve prices is large enough, this benefit of the

FPA may even percolate to higher types as well. A lower reserve price in some ways diminishes competitive pressure amongst bidders.

The paper's conclusions are potentially important for several reasons. First, it demonstrates that it is important to account for the endogeneity of reserve prices when comparing different auction formats. Second, once this is taken into account, it is possible that all parties agree what the preferred auction format is, meaning that there is less of a conflict between revenue and efficiency. Third, there are implications for regulation as well. In the current paper, the self-interested seller is motivated only by profit yet nevertheless often self-selects the auction with the higher social surplus. Regulation that dictates that a conditionally efficient auction like the SPA must be used may prove to be counterproductive as the higher endogenous reserve price may undo the otherwise obvious welfare advantages of the SPA.

2 Model

There is one strong bidder and $n_w \geq 1$ weak bidders. A weak bidder has a privately known type in the interval $[0, \overline{v}_w]$ while the strong bidder's privately known type is in the interval $[0, \overline{v}_s]$. It is important for the analysis that $\overline{v}_s > \overline{v}_w > 0$. A bidder's type describes his willingness to pay for the object at auction. Types are independent across bidders. The bidders and the seller are all risk neutral. The seller has no use of the object herself. Hence, she simply seeks to maximize expected revenue.

To construct type distributions, consider some strictly positive, log-concave, and continuously differentiable function $g(v)$ that is defined for all $v \geq 0$. Let $G(v) = \int_0^v g(x)dx$, $v \geq 0$. It is assumed that bidders in group i, $i = s, w$, draw types from the distribution function

$$F_i(v|\overline{v}_i) = \frac{G(v)}{G(\overline{v}_i)}, \quad v \in [0, \overline{v}_i].$$

The model is a version of one of Maskin and Riley's (2000) models, specifically their "stretch" model in which F_s can be thought of as a stretched version of F_w. Alternatively, F_w can be thought of as a truncation of F_s. Given $g(\cdot)$ and some fixed $\overline{v}_w$, the level of asymmetry between the two groups is parameterized by $\overline{v}_s$.

The structure of the model ensures that F_s dominates the distribution F_w in terms of the likelihood-ratio, which in turn implies that F_s dominates F_w in terms of the hazard rate and also in terms of the reverse hazard rate; see Krishna (2002). The assumptions on $g(v)$ imply that both $F_i(v|\overline{v}_i)$ and $1 - F_i(v|\overline{v}_i)$ are log-concave in v; see Bagnoli and Bergstrom (2005). These properties are important in the analysis of asymmetric auctions; see e.g. Maskin and Riley (2000) and Kirkegaard (2012).

3 Optimal reserve prices and efficiency

It is instructive to begin by examining the SPA. The SPA has an equilibrium in weakly dominant strategies in which bids coincide with valuations. Hence, the SPA is conditionally efficient.

There is a qualitative difference between reserve prices above or below $\overline{v}_w$, as the former excludes weak bidders. Low reserve prices stimulate competition among all bidders and are more likely to lead to a sale. However, higher reserve prices may extract more rent from the strong bidder. Thus, there may be a locally optimal reserve price on the interval $[0, \overline{v}_w]$ and another on the interval $[\overline{v}_w, \overline{v}_s]$. Hence, these local solutions must be compared to find the globally optimal reserve price.

It is intuitive that the more pronounced the asymmetry is, or the higher $\overline{v}_s$ is compared to $\overline{v}_w$, the more likely it is to be optimal to focus on the strong bidder. Thus, a small reserve price is optimal when $\overline{v}_s$ is close to $\overline{v}_w$, but the reserve price may jump discontinuously as $\overline{v}_s$ increases and the seller's strategy switches to extracting rent solely from the strong bidder. This jump is guaranteed to eventually occur if $G(\cdot)$ is not bounded above because in that case the probability that the strong bidder's type exceeds $\overline{v}_w$ approaches 1 as $\overline{v}_s \to \infty$.

Lemma 1 *Holding fixed $\overline{v}_w$, the SPA has a unique revenue maximizing reserve price for all but at most one value of $\overline{v}_s$, denoted $\overline{v}_s^t$. The optimal reserve price is strictly below (above) $\overline{v}_w$ if $\overline{v}_s$ is strictly below (above) the threshold $\overline{v}_s^t$. If $\overline{v}_s = \overline{v}_s^t$ then there is a revenue maximizing reserve price both below and above $\overline{v}_w$. If $G(\cdot)$ is not bounded above then $\overline{v}_s^t < \infty$.*

Proof. See the Appendix. ■

Next, consider the FPA. Once again, weak bidders are excluded at reserve prices above $\overline{v}_w$. Hence, on this range, the SPA and FPA are equally profitable and the locally optimal reserve price in $[\overline{v}_w, \overline{v}_s]$ coincide in the two auctions. However, the two auctions are not revenue equivalent at reserve prices below $\overline{v}_w$ and there is no reason to believe that the locally optimal reserve prices in $[0, \overline{v}_w]$ are the same. Indeed, it follows from Kirkegaard (2012) that the FPA is strictly more profitable than the SPA for any *fixed* reserve price in $[0, \overline{v}_w)$. Hence, when the SPA has a globally optimal reserve price in $[0, \overline{v}_w)$, then the globally optimal reserve price in the FPA is in $[0, \overline{v}_w)$ and the FPA is strictly more profitable than the SPA. This implies that the FPA with an endogenous reserve price is strictly preferred by the seller when $\overline{v}_s \leq \overline{v}_s^t$.

Proposition 1 *Holding fixed $\overline{v}_w$, the FPA with an endogenous reserve price is strictly more profitable than the SPA with an endogenous reserve price for all $\overline{v}_s \leq \overline{v}_s^t$.*

Proof. In text. ■

The paper's main result comes from studying environments in which $\overline{v}_s^t < \infty$. Then, starting from $\overline{v}_s = \overline{v}_s^t$, the model is perturbed by slightly increasing $\overline{v}_s$ above $\overline{v}_s^t$. Although Proposition 1 no longer applies directly, continuity implies that the FPA remains strictly more profitable than the SPA as long as $\overline{v}_s$ is not increased too much. Thus, the seller strictly prefers the FPA in this case.

At the same time, the increase in $\overline{v}_s$ ensures that there is a unique optimal reserve price in the SPA. This is strictly above $\overline{v}_w$ and weak bidders are therefore excluded from the SPA (Lemma 1). However, when $\overline{v}_s = \overline{v}_s^t$, any optimal reserve price in the FPA is strictly below $\overline{v}_w$.[1] This property does not change with a small increase in $\overline{v}_s$. Hence, the reserve price is small enough that the weak bidders are active in the FPA. Consequently, gains from trade are realized more often in the FPA. In fact, it will now be shown that all bidders strictly prefer the FPA to the SPA ex ante.

First, weak bidders weakly prefer the FPA regardless of their types. After all, they are excluded from the SPA but given a chance of winning the FPA if their type is high enough. Thus, weak bidders with high types strictly prefer the FPA.

Second, the strong bidder also weakly prefers the FPA regardless of his type. The argument is the same as above for types that are excluded from the SPA but included in the FPA. Types that are included in both auctions also strictly prefer the FPA. In the SPA, the high reserve price means that it would require a bid somewhere above $\overline{v}_w$ to win. In the FPA, the reserve price is lower but there is competition from the weak bidders. However, rationality on the part of weak bidders implies that they will never bid above $\overline{v}_w$. Thus, the strong bidder can win the FPA with probability one simply by bidding $\overline{v}_w$. Hence, his options are strictly better in the FPA.

[1] Whether there is a unique optimal reserve price in the FPA is unimportant for the argument.

Thus, bidders weakly or strictly prefer the FPA to the SPA at the interim stage, i.e. after types are revealed to bidders but before bidding takes place. It follows that the FPA is strictly preferred to the SPA by bidders at the ex ante stage, i.e. before types are known. As mentioned already, the seller also strictly prefers the FPA at the ex ante stage when $\overline{v}_s$ is slightly above $\overline{v}_s^t$.

Proposition 2 *Holding fixed $\overline{v}_w$, the FPA with an endogenous reserve price ex ante strictly Pareto dominates the SPA with an endogenous reserve price for a set of $\overline{v}_s$ that is strictly above $\overline{v}_s^t$.*

Proof. In text. ■

The next example illustrates Proposition 2. It also shows that the FPA may Pareto dominate the SPA when $\overline{v}_s$ is below $\overline{v}_s^t$. Indeed, the range for which the FPA generates higher social surplus than the SPA can be characterized.

Example 1 (uniform distributions): Assume that $g(v) = 1$ for all $v \geq 0$, implying that distributions are uniform. Fix $\overline{v}_w = 1$ and assume that $n_w = 1$. In the SPA, the two candidates for optimal reserve prices are $\min\{\frac{\overline{v}_s+1}{4}, 1\} \in [0, 1]$ and $\max\{\frac{1}{2}\overline{v}_s, 1\} \in [1, \overline{v}_s]$. Both are interior on their respective intervals if $\overline{v}_s \in (2, 3)$. It can be verified that the higher reserve price is optimal if and only if $\overline{v}_s > 2.408 \equiv \overline{v}_s^t$.

Kaplan and Zamir (2012) derive inverse bidding functions for the FPA with uniform distributions, while allowing for arbitrary reserve prices. Thus, building on their characterization, numerical methods can be used to solve for the optimal reserve price. The optimal reserve price is below 1 as long as $\overline{v}_s < 2.546$. Hence, the conclusion in Proposition 2 holds as long as $\overline{v}_s \in (2.408, 2.546)$. If $\overline{v}_s > 2.546$, then the two auctions share the same high reserve price and are payoff equivalent.

For $\overline{v}_s < 2.408$, both auctions feature reserve prices below 1, but it remains the case that the FPA has the lower reserve price. However, the strong bidder's types near $\overline{v}_s$ prefer the SPA because the weak bidder bids fairly aggressively in the FPA. Hence, the interim ranking is sensitive to the strong bidder's type. Thus, the next step is to (numerically) calculate ex ante expected utility. This establishes that the FPA is an ex ante Pareto improvement over the SPA for all $\overline{v}_s \in (2.36, 2.546)$.

For $\overline{v}_s < 2.36$, the strong bidder prefers the SPA ex ante whereas the weak bidder and the seller prefer the FPA. Nevertheless, total surplus is larger in the FPA than in the SPA as long as $\overline{v}_s \in (1.45, 2.546)$. Hence, the FPA is more efficient than the SPA on a large part of the parameter space once the reserve price is endogenized. ▲

4 Discussion and conclusion

This paper challenges and sometimes overturns the conventional wisdom that the SPA is more efficient than the FPA when bidders are asymmetric. This new insight stems from the fact that the optimal reserve price in the FPA may be lower than in the SPA. The FPA is then more likely to realize gains from trade.

A version of Maskin and Riley's (2000) stretch model is used to describe bidder asymmetry. This model comes with a convenient way of parameterizing the level of asymmetry, by comparing $\overline{v}_s$ to $\overline{v}_w$ or asking how stretched the strong bidder's distribution is. The main argument then follows from perturbing the parameter $\overline{v}_s$ around a critical value. Kirkegaard (2012) shows that the FPA is strictly more profitable than the SPA for all low reserve prices in a larger class of environments. Since this is the critical part of the argument that leads to Proposition 2, the conclusion that the FPA may be more efficient than the SPA should also hold in some settings outside the stretch model, but the perturbation required to formally prove the result is less parsimonious. On the other hand, it is clear that the FPA is not always more efficient than the SPA. For instance, in the context of the stretch model, Example 1 demonstrates that the level of asymmetry must in some sense be large enough for the FPA to dominate. In this vein, Maskin and Riley (2000) and Kirkegaard (2012) also present a class of environments in which the seller prefers the SPA to the FPA, and this holds even with endogenous reserve prices.

The assumption that $\overline{v}_s > \overline{v}_w$ greatly simplifies the arguments that prove the potential superiority of the FPA. However, as Li and Riley (2007) point out, any model with $\overline{v}_s > \overline{v}_w$ can be approximated arbitrarily closely by a model in which F_w is replaced by another distribution that assigns arbitrarily small density to types in $[\overline{v}_w, \overline{v}_s]$. Thus, the $\overline{v}_s > \overline{v}_w$ assumption is convenient but not critical. Similarly, Kirkegaard's (2012) model and results allow for environments where the lowest type of the weak bidders is smaller than the lowest type of the strong bidder.

The main result relies on optimal reserve prices that are so far apart in the two auctions that the weak bidders are excluded from the SPA but not the FPA. However, as in Example 1, there is reason to believe that the reserve price is lower in the FPA than in the SPA even when it is optimal to include the weak bidders in both auctions. The optimal reserve price in the SPA serves a single role: To enforce the optimal amount of rationing. In the FPA, the reserve price has an additional, more indirect, role. When the reserve price is lowered in such an auction, the interaction between types that would have bid above the old reserve price changes too. The lower reserve price causes stronger bidders to lower their guard. Emboldened, the weak bidders take advantage by bidding relatively more aggressively. As a consequence, it is more likely that a weak bidder wins the auction. Hence, lowering the reserve price favors weak bidders at the expense of strong bidders, which tends to be profitable. Intuitively, this extra indirect effect drives the reserve price lower in the FPA. The working paper version of this paper, Kirkegaard (2021), explores this mechanism in more detail and formalizes the intuition. It also explains Example 1 in more detail and discusses auctions with several strong bidders.

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Appendix: Omitted Proof

Proof of Lemma 1. Let $ER^{SPA}(r|\overline{v}_s)$ denote the expected revenue from a SPA with reserve price r. From Myerson (1981), expected revenue is the expected value of the winner's virtual valuation. Here, a bidder in group i, $i = s, w$, with type $v \in [0, \overline{v}_i]$ has virtual valuation

$$J_i(v|\overline{v}_i) = v - \frac{1 - F_i(v|\overline{v}_i)}{f_i(v|\overline{v}_i)}$$

where $f_i(v|\overline{v}_i)$ is the density. Simplifying yields

$$J_i(v|\overline{v}_i) = v - \frac{G(\overline{v}_i) - G(v)}{g(v)}.$$

Note that the weak bidders have higher virtual valuations than the strong bidder for the same types, i.e. $J_w(v|\overline{v}_w) > J_s(v|\overline{v}_s)$ for all $v \in [0, \overline{v}_w]$. Moreover, virtual valuations are strictly increasing in v since $1 - F_i(v|\overline{v}_i)$ is log-concave in v.

Since the SPA is conditionally efficient, by Myerson's (1981) logic it holds that

$$\begin{aligned} ER^{SPA}(r|\overline{v}_s) &= n_w \int_r^{\overline{v}_w} J_w(v|\overline{v}_w) F_w(v|\overline{v}_w)^{n_w - 1} F_s(v|\overline{v}_s) f_w(v|\overline{v}_w) dv \\ &\quad + \int_r^{\overline{v}_s} J_s(v|\overline{v}_s) F_w(\min\{v, \overline{v}_w\}|\overline{v}_w)^{n_w} f_s(v|\overline{v}_s) dv \end{aligned}$$

when $r \in [0, \overline{v}_w]$. The derivative with respect to r can be written as

$$\frac{\partial ER^{SPA}(r|\overline{v}_s)}{\partial r} = -\frac{G(r)^{n_w} g(r)}{G(\overline{v}_s) G(\overline{v}_w)^{n_w}} \left[n_w J_w(r|\overline{v}_w) + J_s(r|\overline{v}_s) \right]. \tag{1}$$

The optimal reserve price must be strictly positive since virtual valuations are strictly negative at $v = 0$. Indeed, since virtual valuations are strictly increasing, $ER^{SPA}(r|\overline{v}_s)$ is either strictly increasing or single-peaked in r. Hence, there is a unique optimal reserve price among reserve prices in $[0, \overline{v}_w]$. Likewise, since $J_s(v|\overline{v}_s)$ is strictly decreasing in $\overline{v}_s$, the optimal reserve price in $[0, \overline{v}_w]$ is non-decreasing in $\overline{v}_s$ (it could be constant at the $\overline{v}_w$ corner).

Consider next reserve prices in the interval $[\overline{v}_w, \overline{v}_s]$, where only the strong bidders are active. The optimal reserve price must be strictly below $\overline{v}_s$ because a reserve price of $\overline{v}_s$ yields zero revenue. Moreover, the same style of arguments as before – but deleting the weak bidders from the analysis – can be applied. Here, $ER^{SPA}(r|\overline{v}_s)$ is either strictly decreasing or single-peaked on the interval $[\overline{v}_w, \overline{v}_s]$ and the solution is non-decreasing in $\overline{v}_s$ (again, it could be constant at the $\overline{v}_w$ corner).

In summary, there is a unique solution on the (compact) interval $[0, \overline{v}_w]$ and one on the interval $[\overline{v}_w, \overline{v}_s]$. Hence, these local solutions must ultimately be compared. Two preliminary observations are provided first.

First, it can never be globally optimal to choose a reserve price of exactly $\overline{v}_w$. Comparing the one-sided limits of the derivative of expected revenue as r approaches $\overline{v}_w$ yields

$$\lim_{r \nearrow \overline{v}_w} \frac{\partial ER^{SPA}(r|\overline{v}_s)}{\partial r} = -\frac{g(\overline{v}_w)}{G(\overline{v}_s)} [n_w \overline{v}_w + J_s(\overline{v}_w|\overline{v}_s)] < -\frac{g(\overline{v}_w)}{G(\overline{v}_s)} J_s(\overline{v}_w|\overline{v}_s) = \lim_{r \searrow \overline{v}_w} \frac{\partial ER^{SPA}(r|\overline{v}_s)}{\partial r}$$

Thus, $ER^{SPA}(r|\overline{v}_s)$ has a kink at $r = \overline{v}_w$ but since the slope is smaller on the left than on the right, it can never achieve a maximum at exactly $r = \overline{v}_w$. This also means that the solutions on $[0, \overline{v}_w]$ and on $[\overline{v}_w, \overline{v}_s]$ do not coincide but are distinct. Recall that both solutions are non-decreasing in $\overline{v}_s$.

Second, when $\overline{v}_s$ is close to $\overline{v}_w$ it must hold that $J_s(v|\overline{v}_s) > 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$. This implies that the globally optimal reserve price must be strictly below $\overline{v}_w$. This is intuitive because the two groups of bidders are almost symmetric when $\overline{v}_s$ is close to $\overline{v}_w$. On the other hand, the optimal reserve price may exceed $\overline{v}_w$ when $\overline{v}_s$ is large enough. For this to occur, it is sufficient that $n_w \overline{v}_w + J_s(\overline{v}_w|\overline{v}_s) \leq 0$ – implying that $ER^{SPA}(r|\overline{v}_s)$ is strictly increasing in r on the interval $[0, \overline{v}_w)$ – which is automatic if $G(\overline{v}_s)$ is large enough. It follows that if G is unbounded, then there are large values of $\overline{v}_s$ for which the optimal reserve price is above $\overline{v}_w$.

Now put these preliminary observations together. The fact that $r = \overline{v}_w$ is never optimal implies that as $\overline{v}_s$ increases, the globally optimal reserve price may discontinuously jump from one interval to the other. If this occurs, the increase in $\overline{v}_s$ causes the optimal reserve price to jump upwards. This is intuitive, as it becomes increasingly more attractive to focus on extracting rent from strong bidders the stronger they get. The assertion is proven next.

Let $ER_H^{SPA}(\overline{v}_s)$ denote the highest expected revenue if the seller is restricted to high reserve prices, $r \geq \overline{v}_w$, and let $ER_L^{SPA}(\overline{v}_s)$ denote the highest expected revenue if the seller is restricted to low reserve prices, $r \leq \overline{v}_w$. The Lemma is trivial if there is no $\overline{v}_s$ for which $ER_H^{SPA}(\overline{v}_s) > ER_L^{SPA}(\overline{v}_s)$. Thus, assume in the remainder that there exists some $\overline{v}_s$ for which $ER_H^{SPA}(\overline{v}_s) > ER_L^{SPA}(\overline{v}_s)$. By a previous argument, this is the case if for example $G(v)$ is not bounded above. Recall that a low reserve price is always optimal, or $ER_H^{SPA}(\overline{v}_s) < ER_L^{SPA}(\overline{v}_s)$, if $\overline{v}_s$ is close enough to $\overline{v}_w$.

As alluded to above, $ER^{SPA}(r|\overline{v}_s)$ is strictly increasing in r on the interval $[0, \overline{v}_w)$ when $n_w\overline{v}_w + J_s(\overline{v}_w|\overline{v}_s) \leq 0$. Since $J_s(\overline{v}_w|\overline{v}_s)$ is strictly decreasing in $\overline{v}_s$, once $n_w\overline{v}_w + J_s(\overline{v}_w|\overline{v}_s) < 0$ then this is still the case as $\overline{v}_s$ increases further. Thus, the globally optimal reserve price is and remains strictly larger than $\overline{v}_w$. The set of $\overline{v}_s$ for which $ER^{SPA}(r|\overline{v}_s)$ has a peak on the interval $(0, \overline{v}_w)$, or $n_w\overline{v}_w + J_s(\overline{v}_w|\overline{v}_s) > 0$, remains of concern. In this case, by continuity of each of the two problems, there must be some value of v_s, denoted $\overline{v}_s^t$, for which $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$. As $r = \overline{v}_w$ cannot be optimal, $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$ can only occur if there is an interior solution in $(0, \overline{v}_w)$ and another in $(\overline{v}_w, \overline{v}_s)$.

Now fix $\overline{v}_s$ and let $r_L \in (0, \overline{v}_w)$ denote the optimal low reserve price and let $r_H \in (\overline{v}_w, \overline{v}_s)$ denote the optimal high reserve price. These are unique, as explained after (1). Using the Envelope Theorem, it can be verified that

$$\begin{aligned}
\frac{\partial ER_H^{SPA}(\overline{v}_s)}{\partial \overline{v}_s} &= n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_H^{SPA}(\overline{v}_s) - \int_{r_H}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right) \\
&> n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_H^{SPA}(\overline{v}_s) - \int_{\overline{v}_w}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right)
\end{aligned}$$

and that

$$\begin{aligned}
\frac{\partial ER_L^{SPA}(\overline{v}_s)}{\partial \overline{v}_s} &= n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_L^{SPA}(\overline{v}_s) - \int_{r_L}^{\overline{v}_s} \left(\frac{G\left(\min\{x, \overline{v}_w\}\right)}{G(\overline{v}_w)} \right)^{n_w} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right) \\
&< n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_L^{SPA}(\overline{v}_s) - \int_{\overline{v}_w}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right).
\end{aligned}$$

However, at $\overline{v}_s = \overline{v}_s^t$, $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$, and it must therefore hold that

$$\frac{\partial ER_H^{SPA}(\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_s^t} > \frac{\partial ER_L^{SPA}(\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_s^t}.$$

This proves that once $\overline{v}_s$ has reached $\overline{v}_s^t$, another small increase in $\overline{v}_s$ unambiguously makes high reserve prices optimal. Thus, the globally optimal reserve price is strictly below $\overline{v}_w$ when $\overline{v}_s < \overline{v}_s^t$ and strictly above $\overline{v}_w$ when $\overline{v}_s > \overline{v}_s^t$. Likewise, there is a unique value of $\overline{v}_s^t$ for which $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$. ■