Efficiency in Asymmetric Auctions with Endogenous Reserve Prices*

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Abstract

In the standard independent private values model, the second-price auction (SPA) is generally taken to be more efficient than the first-price auction (FPA) when bidders are asymmetric. However, this conclusion assumes that reserve prices are identical across auctions. This paper endogenizes the reserve price and shows that it may be lower in the FPA. Hence, gains from trade are realized more often in the FPA. This effect may make the FPA more efficient than the SPA. Indeed, the FPA may Pareto dominate the SPA. That is, the FPA may be more profitable and yet be preferred by all bidders.

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1 Introduction

A central tenet of auction theory is that all commonly used auctions are equally profitable and equally efficient when bidders are symmetric, risk neutral, and have independent private values. This paper concentrates on the symmetry assumption. Even with asymmetric bidders, the second-price auction (SPA) allocates the object efficiently whenever it is sold. In other words, the SPA is *conditionally efficient*: Conditional on a sale, gains from trade are maximized. Since the first-price auction (FPA) does not have this property it is tempting to conclude that the SPA is more efficient than the FPA. Indeed, most of the theoretical and empirical literature studies the apparently more intricate question of which auction is more profitable. Nevertheless, the efficiency question deserves more attention. Not only is efficiency and distributional concerns relevant to society and governments alike, they also inform regulation as well.

As Hu, Matthews, and Zou (2010) among many others point out, in any given auction the possibility that the object may not be traded due for example to a reserve price leads to efficiency loss on its own. It is for this reason that the efficiency question cannot immediately be put to rest. After all, focusing only on conditional efficiency ignores the possibility that the object may not be sold with the same probability in the two auctions. In either auction, the object is sold if and only if there is at least one bidder whose valuation exceeds the reserve price. Hence, if the reserve price is different in the SPA and the FPA then gains from trade are not realized equally often.

This paper considers a seller who designs the reserve price to maximize expected revenue without regard to efficiency. The resulting reserve price may be lower in the FPA than in the SPA, in which case the FPA generates gains from trade more often. Hence, it is no longer obvious which auction is more efficient. If the reserve price is much lower in the FPA than in the SPA, the former may be more efficient.

In fact, it turns out that the FPA may even Pareto dominate the SPA ex ante. That is, the seller and all the bidders, be they strong or weak, may agree that the FPA is preferred. This should be seen in light of the common assertion, originally due to the seminal paper by Maskin and Riley (2000), that the SPA is preferred by strong bidders. The reason is that the FPA tends to favor the weak bidders to the detriment of the strong bidders. However, when the reserve price is lowered it benefits types that would otherwise have been excluded, even if those types belong to strong bidders. If the difference between reserve prices is large enough, this benefit of the FPA may even percolate to higher types as well. A lower reserve price in some ways diminishes competitive pressure amongst bidders.

The optimal reserve price in the SPA serves a single role: To enforce the optimal amount of rationing. In the FPA, the reserve price has an additional, more indirect, role. When the reserve price is lowered in such an auction, the interaction between types that would have bid above the old reserve price changes too. The lower reserve price causes stronger bidders to lower their guard. Emboldened, the weak bidders take advantage by bidding relatively more aggressively. As a consequence, it is more likely that a weak bidder wins the auction. Hence, lowering the reserve price favors weak bidders at the expense of strong bidders, which tends to be profitable. It is this extra indirect effect in the model that drives the reserve price lower in the FPA.

The paper's conclusions are potentially important for several reasons. First, it demonstrates that it is important to account for the endogeneity of reserve prices when comparing different auction formats. Second, once this is taken into account, it is possible that all parties agree what the preferred auction format is, meaning that there is less of a conflict between revenue and efficiency. Third, as discussed in the next section, the impact of the auction format on entry is more subtle than previously thought. Fourth, there are implications for regulation as well. In the current paper, the self-interested seller is motivated only by profit yet nevertheless often self-selects the auction with the higher social surplus. Regulation that dictates that a conditionally efficient auction like the SPA must be used may prove to be counterproductive as the higher endogenous reserve price may undo the otherwise obvious welfare advantages of the SPA.

2 Related literature

Endogenous reserve prices: Myerson (1981) has characterized the optimal mechanism in the independent private values model with risk neutral agents. When bidders are symmetric, and subject to a now standard regularity condition, any commonly used auction implements the optimal mechanism as long as the reserve price is chosen correctly. Thus, the SPA and FPA are equally profitable and efficient in this case.[1]

[1] Kotowski (2018) shows that identity-dependent reserve prices may be profitable in the FPA when the regularity assumption is violated. Milgrom and Weber (1982) shows that the English

Hu, Matthews, and Zou (2010) relax the risk neutrality assumption by allowing agents to be risk averse.[2] Regardless of whether it is the seller or the bidders that are risk averse, the optimal reserve price is lower in the FPA than in the SPA. Hence, the object is more likely to be allocated efficiently in the FPA. Likewise, the seller and the bidders prefer the FPA to the SPA.[3] The current paper instead relaxes the symmetry assumption. The consequences of asymmetry are less clear cut but there are cases where the results mirror those in Hu, Matthews, and Zou (2010).

Empirical papers such as Haile and Tamer (2003), Aradillas-López, Gandhi, and Quint (2013), and Coey, Larsen, and Sweeney (2019) estimate the revenue maximizing reserve price in ascending timber auctions with and without independent values. Coey et al (2017) allow for asymmetric bidders. However, these papers only consider one auction format.

Liu et al (2019) also assume that bidders are symmetric but they assume that the seller has limited commitment power and tries to sell the object at a later date if the reserve price is not met today. As the time between periods vanishes, it is generally optimal to sell the object immediately, implying that no reserve price is used.

Auction design, entry, and efficiency: Auction design is doubly important when the set of bidders is endogenous. Optimal auction design with symmetric and asymmetric bidders are analyzed by Levin and Smith (1994) and Jehiel and Lamy (2015), respectively, when entry costs are fixed and bidders are uninformed about their valuations at the time of entry. Here, efficient auctions with zero reserve prices are optimal when the number of potential entrants is so large that entry probabilities are determined by a zero-profit condition. Gentry, Li, and Lu (2017) show that reserve prices should be used when bidders are partially informed when they enter. Lu and Ye (2013) allow entry costs to be private information. The optimal design consists of a mechanism that allocates entry permits combined with an efficient post-entry auction.

auction and the SPA are more profitable than the FPA for any given reserve price when valuations are affiliated rather than independent. Cai, Riley, and Ye (2007) explore the signaling role of endogeneous reserve prices in a SPA in which the seller has information that is relevant to bidders.

[2] Rosenkranz and Schmitz (2007) consider bidders with reference-based utility. The reserve price affects the reference point. In their setting, the FPA and SPA remain revenue equivalent and the optimal reserve price is the same in either auction.

[3] Hu, Matthews, and Zou (2019) allow for both risk aversion and values that are interdependent. In this case, the optimal reserve price in the SPA may be below the seller's own-use valuations. See also Levin and Smith (1996).

With asymmetric bidders, the literature often focuses on how to entice weak bidders to enter the auction. For instance, in Athey, Levin, and Seira (2011) and Athey, Coey, and Levin (2013), strong bidders enter with probability one in equilibrium, whereas weak bidders randomize. Given strong bidders enter for sure, the problem trivially is to encourage weak bidders to participate. Klemperer (2002) similarly stresses the importance of attracting weak bidders. The FPA has the advantage that it is more desirable for weak bidders, holding fixed the reserve price. This paper shows that it may also be more desirable for strong bidders, once endogenous differences in reserve prices are accounted for. This may be relevant when strong bidders do not enter with probability one. For instance, in Flambard and Perrigne (2006), Marion (2007), and Krasnokutskaya and Seim (2011) the numbers of both weak and strong bidders at auction fluctuate significantly in the data.

Technically speaking, the current paper holds fixed the set of participants. This is primarily to make it easier to convey the main points, as a first step. However, there are applications where the set of bidders is in fact constant. For instance, Brendstrup and Paarsch (2006) identify exactly seven bidders in their fish auctions. Two of the bidders can be thought of as strong and five as weak, but they all participate in virtually every auction. Hence, fixed and exogenous participation appears to be a good approximation in this case. In their setting, however, the reserve price is tightly regulated by the government and it was in fact never binding in their sample.

On the subject of regulation, Marion (2007) and Krasnokutskaya and Seim (2011) among others examine the type of bid preferences that are awarded to bidders from disadvantaged groups in government procurement auctions. They conduct a counterfactual analysis of the impact of varying the level of bid preferences on indicators such as revenue, entry, efficiency, and the probability that a weak bidder wins. The current paper argues that the reserve price deserves the same level of scrutiny, and that the reserve price also has distributional consequences.

While the FPA and SPA are commonly used, other mechanisms are in use in some specific markets. Larsen (2020) and Larsen and Zhang (2018) compare the combination of auctions and bargaining that is employed in the used-car market with the theoretically most efficient mechanism and quantify the efficiency loss of the former. The division of surplus is also examined.

Revenue ranking of asymmetric auctions: Vickrey (1961) observed that the FPA is not efficient when bidders are asymmetric and indeed showed by example that there is no unambiguous revenue ranking between the SPA and FPA.[4] In his two-bidder example, the valuation of one bidder is commonly known. Maskin and Riley (2000) studied the revenue ranking in more detail, assuming that one bidder is ex ante "weak" and the other is "strong." They considered three different environments. The SPA yields higher revenue than the FPA in one environment but the FPA is revenue superior in the other two environments. These papers ignore reserve prices.[5]

Kirkegaard (2012a) extends Maskin and Riley's (2000) analysis in various directions. Among other extensions, he allows for more weak bidders and permits a reserve price. The revenue ranking is unaffected by these changes in any of Maskin and Riley's (2000) three environments. However, Kirkegaard (2012a) does not explore how the optimal reserve prices compare in the two auctions.

Kirkegaard (2021) notes that the analysis may, perhaps surprisingly, be easier when there are more strong bidders, at least when the level of asymmetry is large enough. Thus, whereas Maskin and Riley's (2000) environments are rigidly structured, Kirkegaard (2021) shows that the revenue ranking may in less structured environments depend on the reserve price if the reserve price is restricted to be the same across auctions. The ranking may also change with the numbers of strong and weak bidders. Endogenizing the reserve price and allowing it to differ across auctions, the profitability ranking may likewise depend on the seller's own-use value.

A remark on the approach taken in the current paper is in order. Asymmetric FPA are notoriously difficult to handle. For instance, closed form solutions for bidding strategies are rarely available. Thus, the paper aims to explain the main intuitive drivers of the results through the simplest possible means. Section 4 demonstrates all of the paper's central conclusions simply by applying existing results in a careful manner. This part relies on the results in Kirkegaard (2012a) but it also extends an example due to Kaplan and Zamir (2012). Section 5 then more rigorously explores when and why the reserve price may be lower in the FPA than in the SPA. This part utilizes the methodology in Kirkegaard (2021) and in some cases combine it with arguments that may be familiar from Mares and Swinkels (2014a,b).

[4] Cantillon (2008) argues that bidder asymmetry is detrimental to revenue in the SPA and FPA but not necessarily in the optimal auction. Deb and Pai (2017) prove that there exists some auction with symmetric or anonymous rules that implements the optimal auction.

[5] Hafalir and Krishna (2008) show that the FPA is more profitable than the SPA when bidders are asymmetric and resale is allowed. Moreover, Hafalir and Krishna (2009) show that resale may actually reduce efficiency in an asymmetric FPA due to speculative bidding and asymmetric information even at the resale stage. These papers also ignore reserve prices.

3 Auctions with strong and weak bidders

Throughout, it is assumed that there are two groups of bidders. There is an exogenous number, $n_w \geq 1$ and $n_s \geq 1$, of weak and strong bidders, respectively. A weak bidder has a privately known type in the interval $[0, \overline{v}_w]$ while a strong bidder's privately known type is in the interval $[0, \overline{v}_s]$. It is important for the analysis that $\overline{v}_s > \overline{v}_w > 0$. A bidder's type describes his willingness to pay for the object at auction. Types are independent from one another but are identically distributed within each group. Bidders are assumed to be risk neutral. The seller is also assumed to be risk neutral and to have no use of the object herself. Hence, she simply seeks to maximize expected revenue. The role of the seller's own-use valuation is examined in Kirkegaard (2021).

To convey the main points, it is sufficient and convenient to assume a very specific relationship between the type distributions in the two groups. For this purpose, consider some strictly positive and continuously differentiable function $g(v)$ that is defined for all $v \geq 0$. Let $G(v) = \int_0^v g(x)dx$, $v \geq 0$. It is assumed that bidders in group i, $i = s, w$, draw types from the distribution function

$$F_i(v|\overline{v}_i) = \frac{G(v)}{G(\overline{v}_i)}, \quad v \in [0, \overline{v}_i]$$

with density

$$f_i(v|\overline{v}_i) = \frac{g(v)}{G(\overline{v}_i)}, \quad v \in [0, \overline{v}_i].$$

This model is a version of one of Maskin and Riley's (2000) models, specifically their "stretch" model in which F_s can be thought of as a stretched version of F_w. Alternatively, F_w can be thought of as a truncation of F_s. This specification is convenient because once $\overline{v}_w$ is held fixed, the level of asymmetry between the two groups is parameterized by $\overline{v}_s$. In particular, bidders are symmetric in the limit as $\overline{v}_s \to \overline{v}_w$, but in this paper the assumption that $\overline{v}_s > \overline{v}_w$ is maintained throughout.

Given $\overline{v}_s > \overline{v}_w$, the distribution F_s dominates the distribution F_w in terms of the likelihood-ratio. Krishna (2002) points out that this is a strong property which among other things implies that F_s dominates F_w in terms of the reverse hazard rate. It is well known that this in turn implies that weak bidders bid more aggressively in the FPA for a given type than strong bidders; see e.g. Maskin and Riley (2000) and Kirkegaard (2021). Thus, weak bidders win the FPA more often than is efficient. In comparison, the SPA has an equilibrium in weakly dominant strategies in which any bidder submits a bid that coincides with his type. Thus, the SPA is conditionally efficient as the bidder with the highest valuation wins.

Likelihood-ratio dominance also implies hazard rate dominance. The significance of this is that weaker bidders have higher "virtual valuations" than strong bidders, for comparable types. Here, a bidder in group i, $i = s, w$, with type $v \in [0, \overline{v}_i]$ has virtual valuation

$$J_i(v|\overline{v}_i) = v - \frac{1 - F_i(v|\overline{v}_i)}{f_i(v|\overline{v}_i)}$$

or

$$J_i(v|\overline{v}_i) = v - \frac{G(\overline{v}_i) - G(v)}{g(v)}.$$

Since weaker bidders have higher virtual valuations, it follows from Myerson (1981) that an optimal auction would favor the weak bidders. Bulow and Roberts (1989) explain the intuition by noting that virtual valuation is comparable to marginal revenue in a monopoly problem. However, this does not on its own prove that the FPA is more profitable than the SPA. After all, it is conceivable that the FPA overdoes it and is too favorable to the weak bidders, to the point that the favoritism becomes counterproductive. Additional assumptions are required in order to be able to prove that the FPA is revenue superior.

For the purposes of the current paper, it will for convenience henceforth be assumed that $g(v)$ is log-concave, or equivalently that $f_i(v|\overline{v}_i)$ is log-concave in v for all $\overline{v}_i$, $i = s, w$. This directly implies that the density cannot increase too quickly, which is crucial in existing proofs. In fact, log-concavity of $g(v)$ implies log-concavity of $G(v)$, which is all that is normally assumed (see below). Here, log-concavity of $g(v)$ is assumed because it also implies that the survival function $1 - F_i(v|\overline{v}_i)$ is log-concave; see Bagnoli and Bergstrom (2005). This has the convenient implication that $J_i(v|\overline{v}_i)$ is strictly increasing in v. Hence, if only one group of bidders was present, both the SPA and the FPA would implement the optimal auction as long as the reserve price, r_i, is carefully selected, with $J_i(r_i|\overline{v}_i) = 0$.

Now, given log-concavity of $G(v)$, Maskin and Riley (2000) show that the FPA is strictly more profitable than the SPA when there is only one bidder in each group, or $n_w = n_s = 1$. Kirkegaard (2012a) shows that this revenue ranking holds for any reserve price below $\overline{v}_w$ – as long as it is the same in both auctions – while also allowing for more weak bidders, or $n_w \geq n_s = 1$. Kirkegaard (2012b, Proposition A6) confirms that the same conclusion holds with more strong bidders, or $n_s \geq 1$, as long as $\overline{v}_s$ is large enough relative to $\overline{v}_w$. Finally, the weak bidders are effectively excluded from the auction if the reserve price is $\overline{v}_w$ or higher, in which case the two auction formats yield the same expected revenue.

The remainder of the paper proceeds as follows. Section 4 starts by characterizing optimal reserve prices in the SPA, which is relatively straightforward. It then more narrowly considers the case with one strong bidder. This makes it possible to quickly demonstrate that the optimal reserve price may be lower in the FPA than in the SPA. The main focus of the section is on the implications for efficiency. However, Section 4 does not build much intuition for why the FPA may have a lower reserve price in the first place. Section 5 pursues an explanation. It turns out that the intuition is in some cases clearer when there are several strong bidders.

4 Optimal reserve prices and efficiency

In the following, $\overline{v}_w$ is fixed but $\overline{v}_s$ is allowed to vary, subject only to the condition that $\overline{v}_s > \overline{v}_w$. Let $ER^{SPA}(r|\overline{v}_s)$ and $ER^{FPA}(r|\overline{v}_s)$ denote the expected revenue from a SPA and FPA, respectively, with reserve price r.

It is instructive to begin by examining the SPA. There is a qualitative difference between reserve prices above or below $\overline{v}_w$, as the former excludes weak bidders. Consider first lower reserve prices, i.e. those that are below $\overline{v}_w$. From Myerson (1981), expected revenue is the expected value of the winner's virtual valuation. That is,

$$\begin{aligned} ER^{SPA}(r|\overline{v}_s) &= n_w \int_r^{\overline{v}_w} J_w(v|\overline{v}_w) F_w(v|\overline{v}_w)^{n_w-1} F_s(v|\overline{v}_s)^{n_s} f_w(v|\overline{v}_w) dv \\ &\quad + n_s \int_r^{\overline{v}_s} J_s(v|\overline{v}_s) F_w(\min\{v, \overline{v}_w\}|\overline{v}_w)^{n_w} F_s(v|\overline{v}_s)^{n_s-1} f_s(v|\overline{v}_s) dv \end{aligned}$$

when $r \in [0, \overline{v}_w]$. Given the functional form that F_i takes, the derivative with respect to r can be written as

$$\frac{\partial ER^{SPA}(r|\overline{v}_s)}{\partial r} = -\frac{G(r)^{n_s+n_w-1} g(r)}{G(\overline{v}_s)^{n_s} G(\overline{v}_w)^{n_w}} \left[n_w J_w(r|\overline{v}_w) + n_s J_s(r|\overline{v}_s) \right]. \tag{1}$$

The optimal reserve price must be strictly positive since virtual valuations are strictly negative at $v = 0$. Indeed, since virtual valuations are strictly increasing in r, $ER^{SPA}(r|\overline{v}_s)$ is either strictly increasing or single-peaked in r. Hence, there is a unique optimal reserve price among reserve prices in $[0, \overline{v}_w]$. Likewise, since $J_s(v|\overline{v}_s)$ is strictly decreasing in $\overline{v}_s$, the optimal reserve price in $[0, \overline{v}_w]$ is non-decreasing in $\overline{v}_s$ (it could be constant at the $\overline{v}_w$ corner).

Consider next reserve prices in the interval $[\overline{v}_w, \overline{v}_s]$, where only the strong bidders are active. The optimal reserve price must be strictly below $\overline{v}_s$ because a reserve price of $\overline{v}_s$ yields zero revenue. Moreover, the same style of arguments as before – but deleting the weak bidders from the analysis – can be applied. Here, $ER^{SPA}(r|\overline{v}_s)$ is either strictly decreasing or single-peaked on the interval $[\overline{v}_w, \overline{v}_s]$ and the solution is non-decreasing in $\overline{v}_s$ (again, it could be constant at the $\overline{v}_w$ corner).

Clearly, there may be a local solution on the interval $[0, \overline{v}_w]$ and another on the interval $[\overline{v}_w, \overline{v}_s]$. Hence, these local solutions must be compared.

EXAMPLE 1 (UNIFORM DISTRIBUTIONS): Assume that $g(v) = 1$ for all $v \geq 0$, implying that distributions are uniform. Assume moreover that $n_w = n_s = 1$. As long as $\overline{v}_s < 3\overline{v}_w$, there is a local maximum in the interval $(0, \overline{v}_w)$, specifically at $r = \frac{\overline{v}_s + \overline{v}_w}{4}$. If $\overline{v}_s > 2\overline{v}_w$, there is a local maximum in $(\overline{v}_w, \overline{v}_s)$, namely at $r = \frac{1}{2}\overline{v}_s$. Thus, if $\overline{v}_s \in (2\overline{v}_w, 3\overline{v}_w)$ then there are two local solutions. It can be verified that the higher reserve price is optimal if and only if $\overline{v}_s > 2.408\overline{v}_w$. For future reference, Figure 1 illustrates for the case where $\overline{v}_w = 1$ and $\overline{v}_s = 2.4$, where the lower reserve price is slightly better than the higher reserve price. ▲

EXAMPLE 2 (EXPONENTIAL DISTRIBUTIONS): Assume here that $g(v) = e^{-\alpha v}$ for all $v \geq 0$, with $\alpha > 0$ (if $\alpha = 0$ then type distributions reduce to the uniform distributions). This implies that $G(v) = \frac{1}{\alpha}(1 - e^{-\alpha v})$ and that distribution functions are truncated exponential distributions. Note that $G(\overline{v}_i)$ is bounded above even as $\overline{v}_i \to \infty$. In this case,

$$J_s(v|\overline{v}_s) = v - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha(\overline{v}_s - v)}$$

is bounded below for any given v. In particular, if $\overline{v}_w \geq \frac{1}{\alpha}$ then, regardless of $\overline{v}_s$, it must hold that $J_s(v|\overline{v}_s) \geq 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$. This means that expected revenue is strictly decreasing in r on this interval and indeed that the optimal reserve price must be strictly below $\overline{v}_w$. The intuition here is that increases in $\overline{v}_s$ does not move much probability mass to higher types. In other words, the strong bidders are only negligibly stronger than the weak bidders. Hence, it is suboptimal to exclude the weak bidders who are, after all, almost identical to the strong bidders. ▲

In Example 1, $G(\overline{v}_i)$ diverges to infinity as $\overline{v}_i \to \infty$. This means that $J_s(\overline{v}_w|\overline{v}_s)$

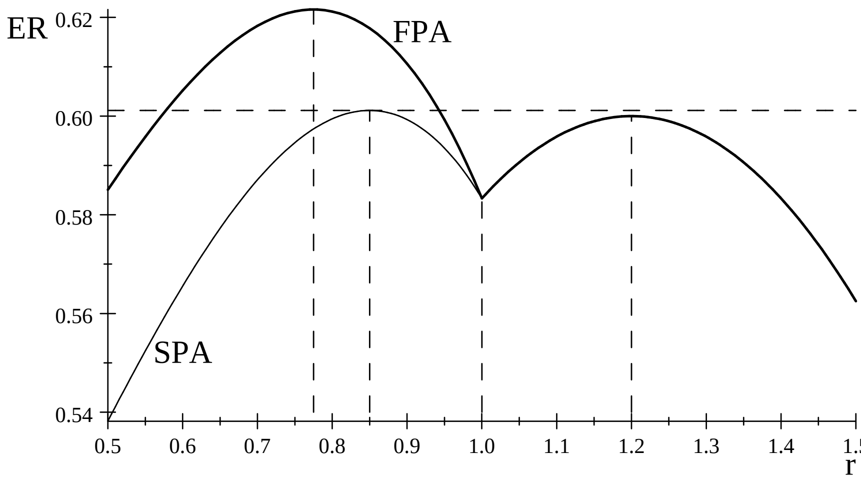


Figure 1: Expected revenue in the SPA and FPA when $g(v) = 1$, $\overline{v}_w = 1$, $\overline{v}_s = 2.4$, and $n_w = n_s = 1$. Expected revenue coincide when $r \geq 1$ but not when $r < 1$.

eventually becomes negative as $\overline{v}_s$ grows, which rules out the situation in Example 2. Note that $G(v)$ is unbounded above if $g(v)$ is weakly increasing or if it is bounded away from zero.

It can never be optimal to choose a reserve price of exactly $\overline{v}_w$. This would necessitate that $ER^{SPA}(r|\overline{v}_s)$ is locally increasing in r as r approaches $\overline{v}_w$ from the left but locally decreasing as r approaches $\overline{v}_w$ from the right. However, it can be verified that these are mutually contradictory properties. Hence, as $\overline{v}_s$ increases, it is possible that the optimal reserve price discontinuously jumps from one interval to another. In Figure 1, the solution jumps from one hump to the other. When and if this occurs, the increase in $\overline{v}_s$ causes the optimal reserve price to jump upwards. This is intuitive, as it becomes increasingly more attractive to focus on extracting rent from strong bidders the stronger they get. In Example 2, the jump never happens and the optimal reserve price is always below $\overline{v}_w$.

Lemma 1 *Holding fixed $\overline{v}_w$ and $n_w, n_s \geq 1$, the SPA has a unique revenue maximizing reserve price for all but at most one value of $\overline{v}_s$, denoted $\overline{v}_s^t(n_s, n_w)$. The optimal reserve price is below (above) $\overline{v}_w$ if $\overline{v}_s$ is strictly below (above) the threshold $\overline{v}_s^t(n_s, n_w)$. If $\overline{v}_s = \overline{v}_s^t(n_s, n_w)$ then there is a revenue maximizing reserve price both below and above $\overline{v}_w$. If $G(v)$ is not bounded above then $\overline{v}_s^t(n_s, n_w) < \infty$. Write $\overline{v}_s^t(n_s, n_w) = \infty$ in the remaining case where the optimal reserve price is below $\overline{v}_w$ for all $\overline{v}_s > \overline{v}_w$.*

Proof. See the Appendix. ■

Next, consider the FPA. Once again, the auction is effectively a symmetric auction at reserve prices above $\overline{v}_w$. Hence, on this range, the SPA and FPA are equally profitable and the locally optimal reserve price in $[\overline{v}_w, \overline{v}_s]$ coincide in the two auctions. However, the two auctions are not revenue equivalent at reserve prices below $\overline{v}_w$ and there is no reason to believe that the locally optimal reserve prices in $[0, \overline{v}_w]$ are the same. If $n_s = 1$, it follows from Kirkegaard (2012a) that the FPA is strictly more profitable than the SPA for any *fixed* reserve price in $[0, \overline{v}_w)$. Hence, when the SPA has a globally optimal reserve price in $(0, \overline{v}_w)$, then the globally optimal reserve price in the FPA is in $(0, \overline{v}_w)$ and the FPA is strictly more profitable than the SPA. Figure 1 illustrates. This implies that the FPA with an endogenous reserve price is strictly preferred by the seller when $\overline{v}_s \leq \overline{v}_s^t(n_s, n_w)$.

Proposition 1 *Assuming that $n_s = 1$ and holding fixed $\overline{v}_w$ and $n_w \geq 1$, the FPA with an endogenous reserve price is strictly more profitable than the SPA with an endogenous reserve price for all $\overline{v}_s \leq \overline{v}_s^t(n_s, n_w)$.*

Proof. In text. ■

The remainder of this section assumes that $n_s = 1$ and that $\overline{v}_s^t(1, n_w) < \infty$. Then, $\overline{v}_s = \overline{v}_s^t(1, n_w)$ is taken as a starting point. Perturbations of $\overline{v}_s$ above and below $\overline{v}_s^t(1, n_w)$ are then considered. These experiments illustrate the main take-aways of the paper. Later, n_s and n_w are allowed to vary but for notational simplicity $\overline{v}_s^t(n_s, n_w)$ is simply written $\overline{v}_s^t$ when no confusion arises as a result.

4.1 Exclusion versus inclusion

Assume that $n_s = 1$. Starting from $\overline{v}_s = \overline{v}_s^t$, a small increase in $\overline{v}_s$ ensures that there is a unique optimal reserve price in the SPA. This reserve price is strictly above $\overline{v}_w$ and excludes weak bidders. However, when $\overline{v}_s = \overline{v}_s^t$, any optimal reserve price in the FPA is strictly below $\overline{v}_w$.[6] This property does not change with a small increase in $\overline{v}_s$. Hence, the optimal reserve price is small enough that the weak bidders are included in the auction. In Figure 1, the small increase in $\overline{v}_s$ causes the right-most hump to become optimal in the SPA whereas the left-most hump remains optimal in the FPA.

[6] It is unknown if there is a unique optimal reserve price in the FPA but this is unimportant for the argument.

Thus, in this environment, it can already be concluded that the FPA is strictly more profitable than the SPA when reserve prices are endogenous and it can likewise be concluded that the FPA has a lower reserve price. This means that gains from trade are realized more often in the FPA. At the same time, weak bidders weakly prefer the FPA to the SPA regardless of their type. After all, they are excluded from the SPA whereas they have a chance of winning the FPA if their type is high enough. The latter types strictly prefers the FPA.

What about the strong bidder? In the SPA, it would take him a bid of at least $r > \overline{v}_w$ to win. In the FPA, the reserve price is lower but on the other hand he faces competition from the weak bidders. Let $\overline{b}_w$ denote the highest possible bid from a weak bidder, i.e. the bid submitted by a weak bidder with type $\overline{v}_w$. Clearly, rationality on the part of weak bidders implies that $\overline{b}_w \leq \overline{v}_w$. Now, the strong bidder can win the auction with probability one simply by bidding $\overline{b}_w$. Hence, his options are better in the FPA. It follows that the strong bidder weakly prefers the FPA regardless of his type and strictly prefers it if his type is above the reserve price in the FPA.

Thus, both bidders weakly or strictly prefer the FPA to the SPA at the interim stage, i.e. after types are revealed to bidders but before bidding takes place. It follows that the FPA is strictly preferred to the SPA by bidders at the ex ante stage, i.e. before types are known. Likewise, the seller strictly prefers the FPA at the ex ante stage. In sum, the FPA is, ex ante, a strict Pareto improvement over the SPA.

Proposition 2 *Assuming that $n_s = 1$ and holding fixed $\overline{v}_w$ and $n_w \geq 1$, the FPA with an endogenous reserve price ex ante strictly Pareto dominates the SPA with an endogenous reserve price for a set of $\overline{v}_s$ that is strictly above $\overline{v}_s^t$.*

Proof. In text. ■

In the setting in Example 1, $\overline{v}_s^t = 2.408$. Using the approach described in the next subsection it can be verified numerically that the conclusion in Proposition 2 holds for $\overline{v}_s \in (2.408, 2.546)$. However, to be clear, if $\overline{v}_s$ increases too far above $\overline{v}_s^t$ then the two auctions share the same high reserve price and are payoff equivalent.

4.2 Accommodating weak bidders

Assume again that $n_s = 1$. Starting from $\overline{v}_s = \overline{v}_s^t$, a small decrease in $\overline{v}_s$ ensures that any optimal reserve price in either auction is strictly below $\overline{v}_w$. Hence, weak bidders compete in either auction. Thus, the two auctions are substantially less different than in the previous subsection. However, it is shown by example that the conclusion in Proposition 2 may nevertheless still hold.

An example is required because it is generally impossible to characterize bidding strategies in closed form in the FPA. This in turn makes it hard to quantify the trade-off that is at the heart of the following analysis in much generality. Nevertheless, the next section attempts to generalize and explain a key insight regarding the relative magnitude of optimal reserve prices in the two auctions.

Example 1 Continued (uniform distributions): As in Example 1, assume that $g(v) = 1$ for all $v \geq 0$ and that $n_w = n_s = 1$. Kaplan and Zamir (2012) derive inverse bidding functions in this case, even allowing for reserve prices. The relevant case is described in their Corollary 1. Let these inverse bidding strategies be denoted $\varphi_s(b)$ and $\varphi_w(b)$ for the strong and weak bidder, respectively. With these in hand, it is possible to infer the distribution of the winning bid and thereby expected revenue. The probability that the winning bid is below b is $F_s(\varphi_s(b)|\overline{v}_s)F_w(\varphi_w(b)|\overline{v}_w) = \frac{\varphi_s(b)}{\overline{v}_s}\frac{\varphi_w(b)}{\overline{v}_w}$. Assume for now that $\overline{v}_w = 1$ and $\overline{v}_s = 2.4$. Then, Figure 1 plots expected revenue as a function of the reserve price. The optimal reserve price in the FPA is $r^{FPA} = 0.775$ whereas it is $r^{SPA} = 0.85$ in the SPA. Once again, the FPA realizes gains from trade more often and is strictly preferred by the seller ex ante.

The FPA is also weakly preferred by the weak bidders at the interim stage. The argument is an extension of the conventional argument that the weak bidder wins more often in the FPA than in the SPA. In this particular case, there is even the additional benefit that the reserve price is lower in the FPA. Hence, the weak bidder strictly prefers the FPA to the SPA ex ante.

Things are more complicated for the strong bidder. Types in (r^{FPA}, r^{SPA}) prefer the FPA as they now have a chance to win. However, it can be shown that types near $\overline{v}_s$ prefer the SPA because the weak bidder is emboldened in the FPA and bids fairly aggressively. Hence, the interim ranking is sensitive to the bidder's type.

However, it is possible to compute ex ante payoff. For each bid b between r^{FPA} and the maximum bid, $\overline{b}$, the strong bidder wins with probability $F_w(\varphi_w(b)|\overline{v}_w)$, in which case he earns payoff of $(\varphi_s(b) - b)$ since his type is $\varphi_s(b)$ and he pays b. Since the density of the strong bidder's bid is $f_s(\varphi_s(b)|\overline{v}_s)\varphi_s'(b)$, ex ante expected utility is

$$\begin{aligned} EU_s^{FPA} &= \int_{r^{FPA}}^{\bar{b}} (\varphi_s(b) - b)\, F_w(\varphi_w(b)|\overline{v}_w) f_s(\varphi_s(b)|\overline{v}_s)\varphi_s'(b)db \\ &= \int_{r^{FPA}}^{\bar{b}} (\varphi_s(b) - b)\, \frac{\varphi_w(b)}{\overline{v}_w} \frac{\varphi_s'(b)}{\overline{v}_s} db, \end{aligned}$$

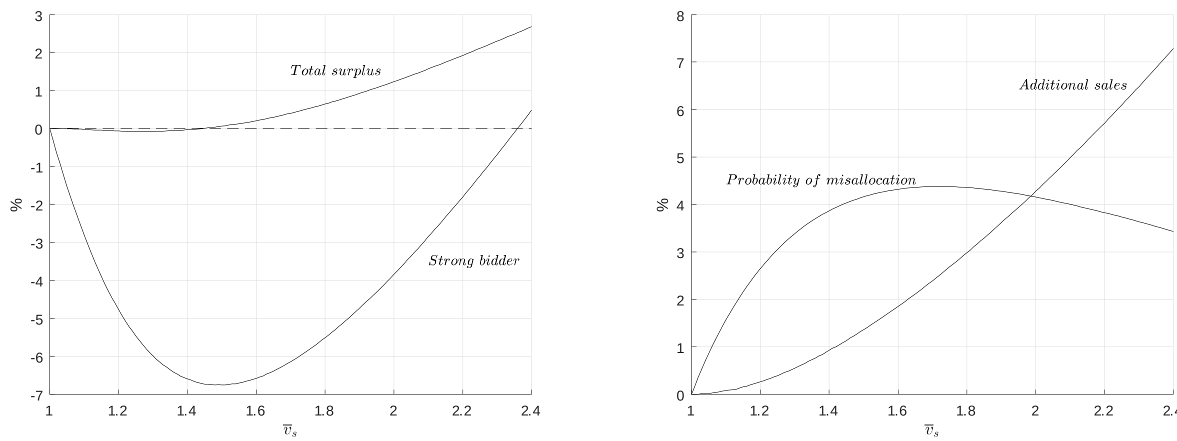
where $\bar{b}$ is the highest possible bid in equilibrium. It can be confirmed numerically that $EU_s^{FPA} = 0.4959$. In comparison, ex ante expected utility in the SPA with the optimal reserve price is $EU_s^{SPA} = 0.4935$. Thus, the strong bidder strictly prefers the FPA to the SPA ex ante, although the difference is small. The weak bidder's expected utility is much lower, and almost negligible, in both auctions. The reason is that the reserve prices are very high relative to even his highest type and that the competitor is so strong. More concretely, expected utility is $EU_w^{FPA} = 0.0109$ and $EU_w^{SPA} = 0.0042$, respectively. Aggregating over both bidders and the seller, total surplus is 1.1284 in the FPA and 1.0989 in the SPA. This is an increase of 2.68%. In conclusion, this example, with $\overline{v}_w = 1$ and $\overline{v}_s = 2.4 < \overline{v}_s^t$, has the same property as in Proposition 2; the FPA strictly Pareto dominates the SPA at the ex ante stage.

Once again, to be clear, it turns out that if $\overline{v}_s$ falls too far below $\overline{v}_s^t$ then the strong bidder prefers the SPA to the FPA ex ante. Hence, the properties in Proposition 2 relies on $\overline{v}_s$ being in an intermediate range. However, the result in Proposition 2 is very strong as it requires all agents to agree that the FPA is better than the SPA. Clearly, the range of $\overline{v}_s$ values for which the FPA generates higher overall social welfare than the SPA is larger than the range for which all agents prefer the FPA.

To demonstrate and quantify these assertions, fix $\overline{v}_w = 1$ but allow $\overline{v}_s$ to vary from 1 to 2.4. For each $\overline{v}_s$ value, the same exercise as above can be carried out. The two panels in Figure 2 summarize the resulting findings. From Figure 2(a), the strong bidder prefers the FPA only when $\overline{v}_s$ is 2.36 or higher.[7] However, the FPA on balance generates larger total surplus for a much wider range of parameters. In particular, this occurs whenever $\overline{v}_s$ is 1.45 or higher. Figure 2(b) illustrates the conflicting forces at play. First, the FPA produces gains from trade up to 7.28% more often than the SPA, due to the lower reserve price. Across all parameter values and both auctions, the optimal reserve price is always above 0.5. This is substantially higher than the

[7] Thus, in combination with the observation after Proposition 2, the FPA Pareto dominates the SPA whenever $\overline{v}_s \in (2.36, 2.55)$.

seller's own-use valuation, which is zero. Thus, whenever an extra sale occurs in the FPA, it contributes significantly to social welfare. On the other hand, the FPA misallocates the object up to 4.38% of the time by allocating it to the weak bidder when the strong bidder has a higher valuation. Some of these instances lead to very significant welfare losses, such as when the weak bidder outbids the strong bidder with a type near $\overline{v}_s$. In many other instances, the weak bidder outbids the strong bidder with only a slightly higher valuation, in which case the welfare loss is negligible.[8] ▲



2(a) Surplus gain from FPA over SPA.

2(b) Extra sales vs. misallocations.

Figure 2: Comparative statics with respect to $\overline{v}_s$ in the uniform model.

4.3 Surplus maximization subject to a revenue target

In the context of Example 1, Figure 1 illustrates that there is a wide range of reserve prices in the FPA that produce expected revenue that exceeds the highest possible expected revenue in the SPA. Given $\overline{v}_s = 2.4$, this range is $[0.5776, 0.9429]$. Hence, the seller does not need to get it "exactly right" for the FPA to be more profitable than the SPA. Indeed, it is possible to lower r far below $r^{FPA} = 0.775$ and still earn higher revenue in the FPA than in the SPA. The lower reserve price further improves the efficiency of the FPA.

These observations lead to the following thought experiment. Imagine an auction run by a government who is concerned about efficiency, yet who needs to achieve a

[8] Finally, there is some degree of over counting of the negative effect of misallocation in the sense that misallocation sometimes occur when no sale would have taken place in the SPA at all. In this case, misallocation between bidders is still better than no sale.

minimum revenue target. The revenue target can be motivated by the desire to avoid distortionary taxation in other parts of the economy to fund government programmes. If the revenue target is very high, then the SPA cannot meet the target but the FPA might be able to. If the target is very low, then the SPA with a zero reserve price is feasible and thus optimal since it maximizes total surplus. Next, consider a revenue target that is about the size of the maximal revenue in the SPA. Then, there is no wiggle room to change the reserve price in the SPA, whereas it can be lowered substantially in the FPA, to the benefit of both bidders. At $r = 0.5776$, total surplus in the FPA increases to 1.1965, which is an improvement of about 8.9% over the SPA with a reserve of 0.775 and the same revenue.

5 Why are reserve prices lower in the FPA?

The environments in Sections 4.1 and 4.2 share the feature that the optimal reserve price is lower in the FPA than in the SPA. The resulting fact that the FPA realizes gains from trade more often is what makes it possible for the FPA to be more efficient than the SPA. Thus, the ranking of reserve prices is central to the paper. So far, however, no attempt has been made to explain the intuition behind this feature. The present section is devoted to explaining why reserve prices are lower in the FPA. The argument relies on insights from mechanism design.

5.1 One strong bidder

It is useful to return to the setting in Figure 1. Figure 3 depicts the type space in this case, with the weak bidder's type on the horizontal axis and the strong bidder's type on the vertical axis. The line marked $t(v)$ denotes the type t of the strong bidder who has the same virtual valuation as the weak bidder with type v, or $J_s(t(v)) = J_w(v)$. Thus, the two bidders are tied in virtual valuation along this line. Virtual valuations are positive on the thick part of $t(v)$ but negative on the thin part of $t(v)$.

The strong bidder has the larger virtual valuation above $t(v)$ and the weak bidder has the highest virtual valuation below $t(v)$. The optimal auction rewards the item to the bidder with the highest virtual valuation, provided it is positive. Virtual valuations are negative in the rectangular area to the south-west of the central point $(0.5, 1.2)$. Hence, an optimal auction withholds the object if the type-profile is in this area. In the rectangular area above it, only the strong bidder has positive virtual valuation and thus wins. The weak bidder wins in the bottom-right area. It is only in the top-right area that both bidders have positive virtual valuation. Then, the strong (weak) bidder wins if the type profile is above (below) the thick part of $t(v)$.

The optimal reserve price in the SPA is $r^{SPA} = 0.85$. In Figure 3, the allocation can be summarized by the thick portion of the 45° line that is marked $s(v)$. This is where the bids (and types) of the two bidders tie. The strong bidder wins above $s(v)$ and the weak bidder wins below $s(v)$, provided the bidder's type is above r^{SPA}. The substantial distance between $s(v)$ and $t(v)$ indicates that the SPA is far from optimal.

Now consider a hybrid auction in which the auction is a FPA but the reserve price is taken from the SPA and thus remains $r = 0.85$. Using Kaplan and Zamir (2012), it can be determined when the strong or weak bidder wins. The curve marked $h(v)$ in Figure 3 represents the dividing line between these events. If the strong bidder has type $h(v)$ then he submits the same bid as the weak bidder with type v. Note that the weak bidder wins more often than is efficient. Sometimes, the weak bidder wins even more often than is optimal, as $h(v)$ is above $t(v)$ when the weak bidder's type is large enough. On balance, however, $h(v)$ is closer to $t(v)$ than $s(v)$ is. This explains why the FPA is more profitable than the SPA for a fixed reserve price.

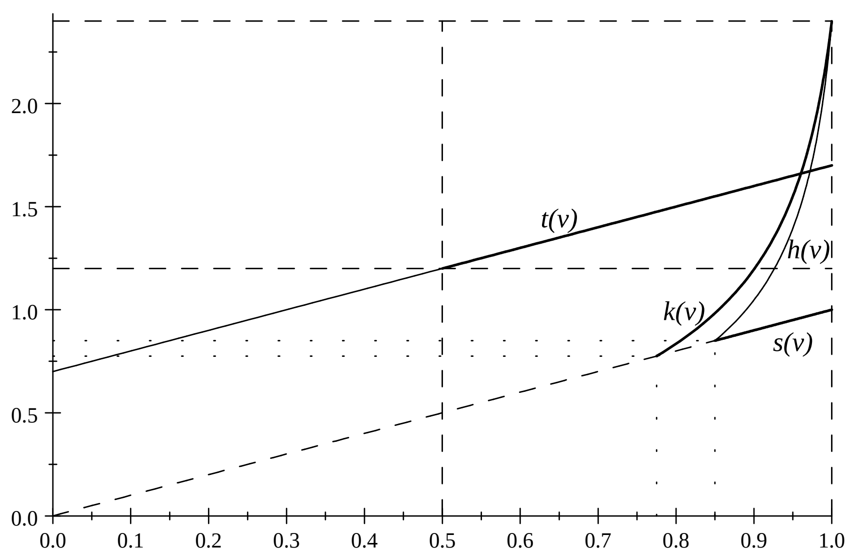


Figure 3: The optimal auction (t), the optimal FPA (k), the optimal SPA (s), and a hybrid auction (h) when $n_s = 1$.

The next question is why the reserve price is lower in the FPA than in the SPA. Lowering the reserve price in the SPA has a direct effect whereas doing so in the FPA has both a direct and an indirect effect. The direct effect from lowering the reserve price is that some types who were previously excluded now have a chance to win. For the weak bidder, these types have positive virtual valuation and the seller benefits from included them. However, the types in question have negative virtual valuations for the strong bidder and their inclusion hurts the seller. These effects compete but at r^{SPA} there is no first-order effect of a marginal change in the reserve price; see (1).

Lowering the reserve price in the FPA has an additional indirect effect on types that would have been included even with the higher reserve price. A lower reserve price makes these types bid lower in the FPA, and more so for the strong bidder. Thus, relatively speaking, the weak bidder becomes more aggressive and now outbids more of the strong bidder's types.

Consider the move from $h(v)$ to $k(v)$ in Figure 3. The latter represents the allocation in the FPA with the optimal reserve price, $r^{FPA} = 0.775$. The weak bidder wins even more often than in the FPA with reserve r^{SPA}. This causes $k(v)$ to move slightly further away from $t(v)$ when the weak bidder's type is high but it also causes $k(v)$ to move much closer to $t(v)$ for a larger set of smaller types. On balance, $k(v)$ is closer to $t(v)$ than $h(v)$ is, and is therefore more profitable. This indirect effect is absent in the SPA and explains why the FPA has a lower reserve price than the SPA.

5.2 More strong bidders

There is an important qualitative difference between a FPA with one strong bidder and a FPA with several strong bidders. In the former, the strong bidder faces competition only from weak bidders. It is for this reason that the strong and weak submit the same bid when they have their respective highest type. In Figure 3, $k(v)$ and $h(v)$ must for this reason terminate at the same point, $(\overline{v}_w, \overline{v}_s)$. This has two pertinent consequences. First, $k(v)$ and $h(v)$ must stay close near the terminal point. Thus, the allocation near the top is relatively insensitive to changes in the reserve price. This is not the case for smaller types, where $k(v)$ and $h(v)$ diverge. In Figure 3, this is what makes a smaller reserve price in the FPA more profitable on balance. Second, $k(\overline{v}_w)$ exceeds $t(\overline{v}_w)$. In other words, a weak bidder with type $\overline{v}_w$ wins too often from a revenue perspective.

Now, there is additional competitive pressure when a strong bidder faces competition from other strong bidders. In particular, it is possible that the strong bidders will spur each other to bid so aggressively that the weak bidders cannot hope to keep up. In this case, a weak bidder with type $\overline{v}_w$ will bid less than a strong bidder with type $\overline{v}_s$. Thus, $k(v)$ may pivot down and terminate at some lower point, $(\overline{v}_w, \widehat{v})$ with $k(\overline{v}_w) = \widehat{v} < \overline{v}_s$. If a strong bidder has type $\widehat{v}$ or higher, he bids so high that he is certain to outbid all the weak bidders.[9]

Note that once $\widehat{v} < \overline{v}_s$, $k(v)$ depends on r but it is unchanged if $\overline{v}_s$ increases further. The technical reason is that the system of differential equations on the range of bids where both groups of bidders are active is unaffected when $\overline{v}_s$ increases, simply because $\frac{f_s(v|\overline{v}_s)}{F_s(v|\overline{v}_s)} = \frac{g(v)}{G(v)}$ is independent of $\overline{v}_s$; see Kirkegaard (2021) for details. Intuitively, it is irrelevant for bidders with lower types that higher types are added to the mix because bidders with low types only win if other bidders have low types too. Stretching the distribution leaves the relevant conditional probabilities unchanged. However, $t(v)$ moves up when $\overline{v}_s$ increases because this causes strong bidders' virtual valuations to decrease. In fact, if $G(v)$ is not bounded above then $t(v)$ increases without bound as $\overline{v}_s$ increases.

Taken far enough, $k(v)$ lies entirely between the 45° line and $t(v)$. Thus, when $G(v)$ is unbounded and $\overline{v}_s$ is large enough, the FPA is unambiguously closer to the optimal auction than the SPA is for a fixed reserve price; see Figure 4. The former therefore yields strictly higher revenue for any given reserve price below $\overline{v}_w$. This argument is originally due to Kirkegaard (2012b, Proposition A6).

Although this is a useful first step, the weakness is that it does not take the endogeneity of the reserve price into account. In particular, it cannot in general be ruled out that the above argument requires $\overline{v}_s$ to be so large that the optimal reserve price in either auction is above $\overline{v}_w$, in which case the argument is of course moot. Thus, there is a need to quantify how large $\overline{v}_s$ needs to be to invoke the argument.

[9] Hubbard and Kirkegaard (2019) is devoted to this issue but their main focus is on determining when this occurs and on how numerical solution methods can be amended to correctly find and simulate equilibrium. Kirkegaard (2021) uses the property to demonstrate that the profit ranking of the FPA and SPA may be more sensitive to such parameters as the composition of bidders and the seller's own-use valuation than previously thought.

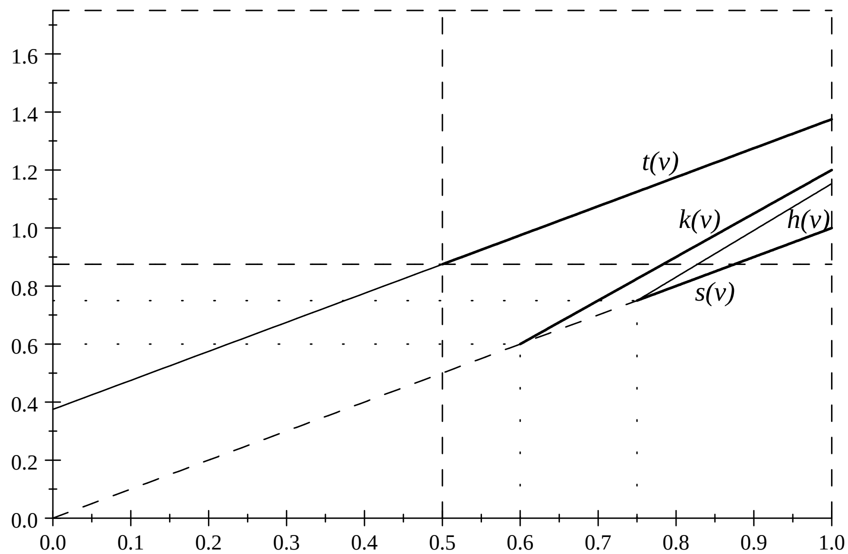


Figure 4: The optimal auction (t), the optimal FPA (k), the optimal SPA (s), and a hybrid auction (h), when $n_s \geq 2$.

It turns out that it is possible to precisely quantify things in the uniform model. Hubbard and Kirkegaard (2019) derive $\widehat{v}$ in closed form in the uniform model when the reserve price is zero. Kirkegaard (2021, Proposition 1) shows that $\widehat{v}$ decreases as the reserve price increases. Thus, in the uniform model, Hubbard and Kirkegaard's (2019) result provides an upper bound on $\widehat{v} = k(\overline{v}_w)$ for any reserve price below $\overline{v}_w$. While the tying-function – unlike $k(v)$ in Figure 3 – cannot be derived explicitly, it can be shown that it is always steeper than $t(v)$. This in turn means that $k(v)$ lies always below $t(v)$ if $k(\overline{v}_w) < t(\overline{v}_w)$, where $t(\overline{v}_w) = \frac{1}{2}(\overline{v}_w + \overline{v}_w)$ in the uniform model.

Lemma 2 *Assume $g(v) = 1$ and that $n_s \geq 2$ and $n_w \geq 1$. Then, for any reserve price $r \in [0, \overline{v}_w)$, it holds that $k(v) < t(v)$ for all $v \in [r, \overline{v}_w]$ if $\overline{v}_s > c(n_s, n_w)\overline{v}_w$, where*

$$c(n_s, n_w) = \frac{\sqrt{(n_s+1)^2(n_s+n_w-1)^2 - 4n_w n_s (n_s+n_w-1)} - (n_s+1)(n_s-1)}{n_w(n_s-1)}.$$

Proof. See the Appendix. ■

Note that $c(n_s, n_w)$ represents a cut-off in the level of asymmetry – as measured by how much $\overline{v}_s$ exceeds $\overline{v}_w$ – above which it is necessarily the case that the FPA is strictly more profitable than the SPA for any reserve price $r \in [0, \overline{v}_w)$. The cut-off is rapidly decreasing in both n_s and n_w. For instance, it falls quickly from $c(2,1) = 1.472$ to $c(3,2) = 1.162$ and on to $c(4,3) = 1.082$ as n_s and n_w both increase. It converges to one as $n_s \to \infty$ and/or $n_w \to \infty$. It is natural to conjecture that the FPA is more profitable than the SPA even if $\overline{v}_s < c(n_s, n_w)\overline{v}_w$ but the proof technique does not make it possible to prove this conjecture.

In the uniform model, $J_s(v|\overline{v}_s) \geq 0$ for all $v \geq \overline{v}_w$ if $\overline{v}_s \leq 2\overline{v}_w$. In this case, the optimal reserve price must necessarily be below $\overline{v}_w$ in both the SPA and the FPA. In other words, $\overline{v}_s^t$ is strictly above $2\overline{v}_w$. As in Example 1, the exact value of $\overline{v}_s^t(n_s, n_w)$ can be computed, although doing so is tedious. For example, $\overline{v}_s^t(2,1) = 2.219\overline{v}_w$ while $\overline{v}_s^t(3,2) = 2.271\overline{v}_w$ and $\overline{v}_s^t(4,3) = 2.287\overline{v}_w$. The conclusion is that there is a relatively large gap between $c(n_s, n_w)\overline{v}_w$ and $\overline{v}_s^t(n_s, n_w)$. For any $\overline{v}_s$ in this range, the FPA with an endogenous reserve price is strictly more profitable than the SPA with an endogenous reserve price.[10]

Figure 4 illustrates for the case where $n_s = 2$, $n_w = 1$, $\overline{v}_w = 1$ and $\overline{v}_s = \frac{7}{4}$. Here, the optimal reserve price in the SPA is $r^{SPA} = \frac{3}{4}$. Extending Hubbard and Kirkegaard's (2019) argument, it can be shown that $\widehat{v} = 1.153$ in a FPA with reserve price $\frac{3}{4}$. In comparison, $t(\overline{v}_w) = 1.375$. Indeed, since Lemma 2 applies, a hybrid FPA auction with reserve price r^{SPA} must lead to a tying-function $h(v)$ that lies always below $t(v)$, as sketched in Figure 4. The point is that $h(v)$ is closer to $t(v)$ than $s(v)$ is. Thus, the FPA is more profitable, holding fixed the reserve price.

The reserve price in the FPA is also no higher than the reserve price in the SPA. As mentioned in the previous subsection, there is no direct first-order effect of a marginal change in the reserve price away from r^{SPA}. However, a reduction in the reserve price has the additional indirect effect in the FPA that the allocation, $k(v)$, moves even closer to $t(v)$. The fact that $k(v)$ lies above $h(v)$ is proven formally in Kirkegaard (2021). Due to this indirect effect, any optimal reserve price in the FPA is no larger than in the SPA, even though its exact value cannot be determined analytically.[11]

Recall that $\overline{v}_s^t > c(n_s, n_w)\overline{v}_w$ in the uniform model. This means that there are $\overline{v}_s$ values slightly above $\overline{v}_s^t$ for which the optimal reserve price in the SPA but not the

[10] Note also that $\overline{v}_s^t$ increases with n_w. The reason is that higher n_w makes reserve prices below $\overline{v}_w$ more profitable but does not change the profitability of reserve prices above $\overline{v}_w$. Hence, the gap between $c(n_s, n_w)\overline{v}_w$ and $\overline{v}_s^t$ increases with n_w.

[11] Due to a technical complication, it is hard to prove that any optimal reserve price in the FPA is strictly smaller than r^{SPA}. See the proof of Proposition 3.

FPA excludes weak bidders. As in Proposition 2, the FPA is once again a Pareto improvement over the SPA. Starting at $r^{SPA} > \overline{v}_w$, the two auctions are payoff equivalent to bidders. Lowering the reserve price can only benefit them. Likewise, the FPA is more profitable, to the benefit of the seller.

The next proposition summarizes this discussion for the uniform model and formally proves the results.

Proposition 3 *Assume $g(v) = 1$ and that $n_s \geq 2$, $n_w \geq 1$ and $\overline{v}_s > c(n_s, n_w)\overline{v}_w$. Then, the following properties hold:*

(i) The FPA with an endogenous reserve price is weakly more profitable than the SPA with an endogenous reserve price and any optimal reserve price in the FPA is weakly smaller than any optimal reserve price in the SPA.

(ii) If the SPA has an optimal reserve price below $\overline{v}_w$ then the FPA is strictly more profitable than the SPA and realizes gains from trade no less often.

(iii) The FPA with an endogenous reserve price ex ante strictly Pareto dominates the SPA with an endogenous reserve price for a set of $\overline{v}_s$ strictly above $\overline{v}_s^t(n_s, n_w)$.

Proof. See the Appendix. ■

Small perturbations of $g(v)$ should not be expected to affect the conclusion of Proposition 3. However, it is possible to more formally generalize Proposition 3 in at least two directions.

First, assume that $g(v)$ is increasing. Then, $F_i(v|\overline{v}_i)$ is convex and it dominates the uniform distribution in terms of the likelihood-ratio. This implies that bidders are more likely to have higher types. Adapting a result in Hubbard and Kirkegaard (2019), it can be proven that the increased competition forces $\widehat{v}$ to drop even lower. At the same time, $t(\overline{v}_w)$ moves up because $J_s(\overline{v}_w)$ diminishes as the strong bidders become stronger, yet $J_w(\overline{v}_w)$ remains equal to $\overline{v}_w$ since virtual valuations and types coincide at the top. Hence, the gap between $t(v)$ and $k(v)$ widens near $v = \overline{v}_w$. This suggests that it remains the case that $k(v)$ is globally below $t(v)$, which is the main ingredient in the arguments surrounding Figure 4. However, to establish this formally, additional technical assumptions are needed. It is sufficient to assume that

$$\frac{d}{dv}\frac{g'(v)v}{g(v)} \geq 0 \text{ for all } v > 0. \tag{2}$$

Recall the assumption that $g(v)$ is log-concave. The assumption in (2) is equivalent to assuming that $g(e^x)$ is log-convex for all $x \in \mathbb{R}$. The two assumptions are not mutually exclusive but taken together they imply that $g(v)$ is increasing, thus making the latter assumption redundant. All the assumptions are satisfied if e.g. $g(v) = v^\alpha$ or $g(v) = e^{\alpha v}$, with $\alpha \geq 0$.[12] The uniform model is a special case, with $\alpha = 0$. The assumption in (2) is stronger than is needed but it is the most succinct way of guaranteeing that $t(v)$ shifts up. Technically, (2) ensures that $t(v)$ is even flatter, and therefore even higher, than in the uniform model. In a different asymmetric auction application with only two bidders, Mares and Swinkels (2014a,b) likewise impose conditions on the primitives that serve to bound the slope of $t(v)$. Some of their results also require densities to be monotone, as is the case here.

Proposition 4 *Assume that (2) is satisfied and that $n_s \geq 2$, $n_w \geq 1$ and $\overline{v}_s > c(n_s, n_w)\overline{v}_w$. Then, (i)–(iii) in Proposition 3 hold if $\overline{v}_s^t(n_s, n_w) > c(n_s, n_w)\overline{v}_w$.*

Proof. See the Appendix. ■

The next example explains the role of the qualifier that $\overline{v}_s^t(n_s, n_w) > c(n_s, n_w)\overline{v}_w$.

Example 3 (power distributions): Given (2), the stronger bidders are now stronger than is the case in the uniform model. Hence, it takes a smaller value of $\overline{v}_s$ to make high reserve prices optimal. In other words, if $g(v)$ increases quickly enough, then it may happen that $c(n_s, n_w)\overline{v}_w > \overline{v}_s^t(n_s, n_w)$. The FPA is trivially no less profitable and no less efficient than the SPA if $\overline{v}_s > c(n_s, n_w)\overline{v}_w > \overline{v}_s^t(n_s, n_w)$. In other words, part (*i*) of Proposition 3 is immediate in this case. However, parts (*ii*) and (*iii*) are more interesting but for these it is required that $\overline{v}_s^t(n_s, n_w) > c(n_s, n_w)\overline{v}_w$.

Recall the sufficient condition that $\overline{v}_s < \overline{v}_s^t(n_s, n_w)$ if $J_s(\overline{v}_w|\overline{v}_s) \geq 0$. If $g(v) = v^\alpha$, with $\alpha \geq 0$, then this is in turn satisfied as long as

$$\overline{v}_s \leq (2+\alpha)^{\frac{1}{1+\alpha}}\,\overline{v}_w.$$

The factor $(2+\alpha)^{\frac{1}{1+\alpha}}$ exceeds $c(n_s, n_w)$ for all (n_s, n_w) if $\alpha \leq 3.3$. It exceeds $c(3,2)$ if $\alpha \leq 19.4$ and $c(4,3)$ if $\alpha \leq 48.85$. In these cases, there is still a gap between $c(n_s, n_w)\overline{v}_w$ and $\overline{v}_s^t(n_s, n_w)$. Since the uniform model corresponds to $\alpha = 0$, it requires a very large move away from the uniform distribution in order to destroy the gap. ▲

[12] Note that $g(e^x)$ is log-linear when $g(v) = v^\alpha$ and that $g(v)$ is log-linear when $g(v) = e^{\alpha v}$.

Propositions 3 and 4 hold the set of bidders fixed and impose conditions on $g(v)$. In either case, the degree of asymmetry cannot be too small or else the technical arguments do not have enough bite. That is, $\overline{v}_s > c(n_s, n_w)\overline{v}_w$ is required. The second approach to generalizing Proposition 3 is to instead fix $g(v)$ and to impose conditions on the set of bidders. This makes it possible to examine auctions with small asymmetries. More formally, think of the asymmetry as being small if $J_s(\overline{v}_w|\overline{v}_s) > 0$. In this case, the optimal reserve price in either auction is strictly below $\overline{v}_w$ regardless of the number of weak and strong bidders that are present at auction.

For fixed values of $\overline{v}_w$ and $\overline{v}_s > \overline{v}_w$, and for any log-concave $g(v)$, $t(v)$ is always bounded away from v. However, Kirkegaard (2021, Proposition 3) shows that $k(v)$ converges to v for all $v \in (r, \overline{v}_w]$ and any $r \in (0, \overline{v}_w)$ when $n_s \to \infty$ and/or $n_w \to \infty$. Intuitively, the FPA is arbitrarily close to efficient with enough competition. Thus, when the set of bidders is large enough, it holds that $s(v) < k(v) < t(v)$ for all $v \in (r, \overline{v}_w]$.

Proposition 5 *Fix $g(\cdot)$ and assume that $\overline{v}_w$ and $\overline{v}_s$ are close enough that $J_s(\overline{v}_w|\overline{v}_s) > 0$. Then, there exists (n_s, n_w) for which the FPA is strictly more profitable and has a weakly lower reserve price than the SPA.*

Proof. In text. ■

5.3 Other distributions

It has been assumed throughout that $\overline{v}_s > \overline{v}_w$. It is natural to worry that this assumption is crucial to the results. However, this is not the case. As Li and Riley (2007) point out, the set-up can be approximated arbitrarily closely by a model in which F_w is replaced by another distribution, $\widetilde{F}_w$, that assigns arbitrarily small density to types in $[\overline{v}_w, \overline{v}_s]$. Then, F_s and $\widetilde{F}_w$ share the same support and the results in Propositions 2–4 still hold.

After the perturbation, F_s is no longer a "stretched" version of F_w. However, the assumption that F_s stretches F_w is not important for the main intuition or for most of the steps in the analysis. Stretching any $F_s(v|\overline{v}_s)$ by increasing $\overline{v}_s$ decreases the strong bidders' virtual valuations and pushes $t(v)$ up, even if F_s bears no relationship to F_w to start. At the same time, when $\overline{v}_s > \widehat{v}$ a further increase in $\overline{v}_s$ does not change the allocation, as summarized by $k(v)$. Hence, the same intuition as outlined in the previous section still applies.

6 Conclusion

This paper challenges and sometimes overturns the conventional wisdom that the SPA is more efficient than the FPA when bidders are asymmetric. The reason is that the optimal reserve price in the FPA may be lower than in the SPA. This means that the FPA is more likely to realize gains from trade. This property was established in settings with one or more strong bidders. In fact, the intuition was in some ways easier to convey when there are several strong bidders and some of the technical results could only be proven for such cases. Along with the companion paper, Kirkegaard (2021), this suggests that there may at times be methodological advantages to studying auctions with more than one strong bidder. Indeed, the empirical literature suggests that this case is more frequent in the real world.

Accounting for the endogeneity of the reserve price is potentially important beyond the SPA and FPA. For instance, future research may determine that it can influence the ranking between various multi-unit auction formats. Baisa and Burkett (2018) study the properties of different multi-unit auctions, including which are preferred by weak and strong bidders, but they abstract away from a reserve price.

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Appendix: Omitted Proofs

Proof of Lemma 1. To start, note that there is a local maximum at some $r > \overline{v}_w$ if and only if $J_s(\overline{v}_w|\overline{v}_s) < 0$. This necessitates that $\overline{v}_s$ is large enough and, as explained after Examples 1 and 2, this is also a sufficient condition if $G(v)$ is not bounded above (Example 2 shows that there are cases where $J_s(\overline{v}_w|\overline{v}_s) < 0$ never happens regardless of how large $\overline{v}_s$ is). Now, it follows from (1) that if $\overline{v}_s$ is such that $ER^{SPA}(r|\overline{v}_s)$ is strictly increasing in r on the interval $[0, \overline{v}_w]$ then this remains the case as $\overline{v}_s$ increases. In these case, $J_s(\overline{v}_w|\overline{v}_s) < 0$ and the optimal reserve price is then strictly larger than $\overline{v}_w$. This remains the case as $\overline{v}_s$ increases. Hence, the interesting case is when $\overline{v}_s$ is so low that $ER^{SPA}(r|\overline{v}_s)$ has a peak in $(0, \overline{v}_w)$.

Let $ER_H^{SPA}(\overline{v}_s)$ denote the highest expected revenue if the seller is restricted to high reserve prices, $r \geq \overline{v}_w$, and let $ER_L^{SPA}(\overline{v}_s)$ denote the highest expected revenue if the seller is restricted to low reserve prices, $r \leq \overline{v}_w$. When $\overline{v}_s$ is small and close to $\overline{v}_w$, $ER_H^{SPA}(\overline{v}_s) < ER_L^{SPA}(\overline{v}_s)$ because in this case $J_s(v|\overline{v}_s) > 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$. This is intuitive because the two groups of bidders are almost symmetric when $\overline{v}_s$ is small. It was argued in the previous paragraph that $ER_H^{SPA}(\overline{v}_s) > ER_L^{SPA}(\overline{v}_s)$ requires that $\overline{v}_s$ is large. The Lemma is trivial if there is no $\overline{v}_s$ for which $ER_H^{SPA}(\overline{v}_s) > ER_L^{SPA}(\overline{v}_s)$; this is the case denoted $\overline{v}_s^t = \infty$. Thus, assume in the remainder that there exists some $\overline{v}_s$ for which $ER_H^{SPA}(\overline{v}_s) > ER_L^{SPA}(\overline{v}_s)$. As mentioned above, this is the case if $G(v)$ is not bounded above.

Then, by continuity of each of the two problems, there must be some value of v_s, denoted $\overline{v}_s^t$, for which $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$; although $\overline{v}_s^t$ depends on n_s and n_w this is suppressed here for simplicity. As $r = \overline{v}_w$ cannot be optimal, $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$ can only occur if there is an interior solution in $(0, \overline{v}_w)$ and another in $(\overline{v}_w, \overline{v}_s)$. To prove the Lemma now requires only that uniqueness of $\overline{v}_s^t$ is established.

Now fix $\overline{v}_s$ and let $r_L \in (0, \overline{v}_w)$ denote the optimal low reserve price and let $r_H \in (\overline{v}_w, \overline{v}_s)$ denote the optimal high reserve price. These are unique, as explained after (1). Using the Envelope Theorem, it can be verified that

$$\begin{aligned}
\frac{\partial ER_H^{SPA}(\overline{v}_s)}{\partial \overline{v}_s} &= n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_H^{SPA}(\overline{v}_s) - \int_{r_H}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right) \\
&> n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_H^{SPA}(\overline{v}_s) - \int_{\overline{v}_w}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right)
\end{aligned}$$

and that

$$\begin{aligned}\frac{\partial ER_L^{SPA}(\overline{v}_s)}{\partial \overline{v}_s} &= n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_L^{SPA}(\overline{v}_s) - \int_{r_L}^{\overline{v}_s} \left(\frac{G\left(\min\{x, \overline{v}_w\}\right)}{G(\overline{v}_w)} \right)^{n_w} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right) \\ &< n_s \frac{g(\overline{v}_s)}{G(\overline{v}_s)} \left(\overline{v}_s - ER_L^{SPA}(\overline{v}_s) - \int_{\overline{v}_w}^{\overline{v}_s} \left(\frac{G(x)}{G(\overline{v}_s)} \right)^{n_s - 1} dx \right).\end{aligned}$$

However, at $\overline{v}_s = \overline{v}_s^t$, $ER_H^{SPA}(\overline{v}_s^t) = ER_L^{SPA}(\overline{v}_s^t)$, and it must therefore hold that

$$\frac{\partial ER_H^{SPA}(\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_s^t} > \frac{\partial ER_L^{SPA}(\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_s^t}.$$

This proves that once $\overline{v}_s$ has reached $\overline{v}_s^t$, another small increase in $\overline{v}_s$ unambiguously makes high reserve prices optimal. Hence, there is a unique value of $\overline{v}_s^t$. ■

Proof of Lemma 2. STEP 1: Comparing the slopes of $k(v)$ and $t(v)$. In the uniform model, virtual valuations take the simple form $J_i(v|\overline{v}_i) = 2v - \overline{v}_i$. Thus, $t(v) = v + \frac{1}{2}(\overline{v}_s - \overline{v}_w)$, with $t'(v) = 1$. Holding fixed the reserve price, r, at some value below $\overline{v}_w$, let $k(v|r)$ denote the type of a strong bidder that submits the same bid as a weak bidder of type v. Kirkegaard (2021, Proposition 1) proves that $k(v|r) > v$ for all $v \in (r, \overline{v}_w)$ and derives an expression for $k'(v|r)$. Given $k(v|r) > v$, $k'(v|r)$ is bounded by

$$k'(v|r) > \frac{F_s(k(v|r)|\overline{v}_s)}{f_s(k(v|r)|\overline{v}_s)} \frac{f_w(v|\overline{v}_w)}{F_w(v|\overline{v}_w)} = \frac{G(k(v|r))}{g(k(v|r))} \frac{g(v)}{G(v)} \geq 1,$$

where the last inequality comes from the fact that $G(v)$ is log-concave and $k(v|r) > v$. Hence, $k'(v|r) > t'(v)$ for all $v \in (r, \overline{v}_w]$.

STEP 2: Comparing the end-points of $k(v)$ and $t(v)$. Hubbard and Kirkegaard (2019, Proposition 5) have proven that when there is no reserve price,

$$k(\overline{v}_w|0) = \min\{\overline{v}_s, q(n_s, n_w)\overline{v}_w\},$$

where

$$q(n_s, n_w) = \frac{\sqrt{(n_s+1)^2(n_s+n_w-1)^2 - 4n_w n_s(n_s+n_w-1)} + 2n_w n_s - (n_s+1)(n_s+n_w-1)}{2n_w(n_s-1)}.$$

In comparison, $t(\overline{v}_w) = \frac{1}{2}(\overline{v}_w + \overline{v}_s)$. Hence, $t(\overline{v}_w) > k(\overline{v}_w|0)$ if and only if

$$\overline{v}_s > (2q(n_s, n_w) - 1)\,\overline{v}_w.$$

It is straightforward to verify that the term in the parenthesis reduced to $c(n_s, n_w)$ in Lemma 2. From Kirkegaard (2021, Proposition 1), $k(v|r)$ is decreasing in r. Hence, $t(\overline{v}_w) > k(\overline{v}_w|r)$ for all $r \in [0, \overline{v}_w)$ if and only if the condition in Lemma 2 holds.

Step 3: A global comparison of $k(v)$ and $t(v)$. From step 2 and given the condition in Lemma 2, $t(\overline{v}_w) > k(\overline{v}_w|r)$ for all $r \in [0, \overline{v}_w)$. From step 1, $k'(v|r) > t'(v)$ for all $v \in (r, \overline{v}_w]$. It now follows that $k(v|r) < t(v)$ for all $v \in [r, \overline{v}_w]$ and for all $r \in [0, \overline{v}_w)$. ■

Proof of Proposition 3. Step 1: Comparing expected revenue. Recall that $ER^{SPA}(r|\overline{v}_s)$ and $ER^{FPA}(r|\overline{v}_s)$ denote expected revenue for some reserve price r in the SPA and FPA respectively. Kirkegaard (2012a) showed that for $r \in [0, \overline{v}_w)$, the revenue difference between the two auctions is

$$\begin{aligned}
\Delta(r|\overline{v}_s) &= ER^{FPA}(r|\overline{v}_s) - ER^{SPA}(r|\overline{v}_s) \\
&= \int_r^{\overline{v}_w} \left(\int_v^{k(v|r)} (J_w(v|\overline{v}_w) - J_s(x|\overline{v}_s))\, dF_s(x|\overline{v}_s)^{n_s} \right) dF_w(v|\overline{v}_w)^{n_w}.
\end{aligned}$$

Since $k(v|r)$ is strictly below $t(v)$ by Lemma 2, all the terms $(J_w(v|\overline{v}_w) - J_s(x|\overline{v}_s))$ are all strictly positive. This confirms that the FPA is strictly more profitable than the SPA for any fixed reserve price below $\overline{v}_w$. The auctions are of course revenue equivalent at fixed reserve prices above $\overline{v}_w$. Hence, if the SPA has an optimal reserve price below $\overline{v}_w$ then so does the FPA and the latter is strictly more profitable. If the SPA has an optimal reserve price above $\overline{v}_w$, then the FPA with an optimal reserve price is no less profitable. This proves the assertions in the proposition that relate to the revenue ranking.

Step 2: Comparing optimal reserve prices. If the optimal reserve price in the SPA is uniquely above $\overline{v}_w$ then the optimal reserve price in the FPA is either the same or it takes a value strictly below $\overline{v}_w$. In the remainder, assume therefore that the SPA has an optimal reserve price below $\overline{v}_w$. Let this be denoted r^{SPA}. It follows from the previous step that any optimal reserve price in the FPA must be below $\overline{v}_w$ as well. Now, for given reserve prices $r' < r < \overline{v}_w$, and suppressing $\overline{v}_w$ and $\overline{v}_s$ for notational simplicity,

$$\Delta(r|\overline{v}_s) - \Delta(r'|\overline{v}_s) = -\int_{r'}^{r} \left(\int_{v}^{k(v|r')} (J_w(v) - J_s(x))\, dF_s(x)^{n_s} \right) dF_w(v)^{n_w} \\ - \int_{r}^{\overline{v}_w} \left(\int_{k(v|r)}^{k(v|r')} (J_w(v) - J_s(x))\, dF_s(x)^{n_s} \right) dF_w(v)^{n_w}.$$

Recall that by Kirkegaard (2021, Proposition 1), $k(v|r') > k(v|r)$. Since $k(v|r')$ is also strictly below $t(v)$, all the terms $(J_w(v) - J_s(x))$ are once again strictly positive. Hence,

$$\Delta(r|\overline{v}_s) - \Delta(r'|\overline{v}_s) < 0.$$

In other words, the revenue difference between the FPA and the SPA become more pronounced when the reserve price is smaller. Figure 1 illustrates this property.[13]

By definition,

$$ER^{FPA}(r|\overline{v}_s) = ER^{SPA}(r|\overline{v}_s) + \Delta(r|\overline{v}_s).$$

Likewise, by definition $ER^{SPA}(r^{SPA}|\overline{v}_s) \geq ER^{SPA}(r|\overline{v}_s)$ for any r (including those above $\overline{v}_w$). It has just been established that $\Delta(r^{SPA}|\overline{v}_s) > \Delta(r|\overline{v}_s)$ for all $r > r^{SPA}$. Now, imagine, by contradiction, that the FPA has an optimal reserve price, r^{FPA}, that is below $\overline{v}_w$ but strictly above r^{SPA}. Then, lowering r from r^{FPA} to r^{SPA} increases both $ER^{SPA}(r|\overline{v}_s)$ and $\Delta(r|\overline{v}_s)$. Thus, $ER^{FPA}(r|\overline{v}_s)$ increases, contradicting the assumption that $r^{FPA} > r^{SPA}$. Consequently, any optimal reserve price in the FPA can be no larger than r^{SPA}.

As a final comment to this step, note that it was not claimed or proven formally that the optimal reserve price in the FPA is *strictly* below the optimal reserve price in the SPA even when the latter is below $\overline{v}_w$. Since $\Delta(r|\overline{v}_s)$ is strictly decreasing it must be differentiable in r almost everywhere. However, this does not rule out that the derivative is zero at some specific r (it cannot have a zero derivative on a proper interval). In this case, $ER^{FPA}(r|\overline{v}_s) = ER^{SPA}(r|\overline{v}_s) + \Delta(r|\overline{v}_s)$ may technically be maximized where $ER^{SPA}(r|\overline{v}_s)$ is maximized. It seems "unlikely" that $ER^{SPA}(r|\overline{v}_s)$ and $\Delta(r|\overline{v}_s)$ happen to have a stationary point at the same place, but it has not been ruled out.

[13] This is a coincidence as the proof does not cover the situation in Figure 1, where $n_s = 1$ rather than $n_s \geq 2$.

STEP 3: Efficiency. A small increase in $\overline{v}_s$ above $\overline{v}_s^t$ means that the unique optimal reserve price in the SPA is above $\overline{v}_w$ whereas any optimal reserve price in the FPA is below $\overline{v}_w$. The seller prefers the FPA, as proven above. Any bidder, weak or strong, strictly prefers the FPA to the SPA if his type is in the interval between the two reserve prices as such a bidder would be excluded from the SPA. This proves that any weak bidder strictly prefers the SPA ex ante. It will next be shown that any strong bidder with a higher type also prefers the FPA. Thus, any strong bidder strictly prefers the FPA ex ante.

Consider first strong bidders with types at or below $\widehat{v}$. All these types strictly prefer the FPA if $\widehat{v} < r^{SPA}$ as they are effectively excluded from the SPA. Thus, assume now that $\widehat{v} \geq r^{SPA}$. Types below r^{SPA} prefer the FPA for the same reason as before, which leaves types in $\left[r^{SPA}, \widehat{v}\right]$ to consider for now. In the FPA a strong bidder with type $\widehat{v}$ submits a bid identical to the bid submitted by a weak bidder with type $\overline{v}_w$, $\overline{b}_w$. In order for $\overline{b}_w$ to be rational to weak bidders, it must hold that $\overline{b}_w \leq \overline{v}_w < r^{SPA}$. Hence, a strong bidder in the FPA can outbid all the weak bidders and all the strong bidders with type below $\widehat{v}$ by submitting a bid below $\overline{v}_w$. In other words, his payoff is at least

$$(v - \overline{v}_w)\, F_s(\widehat{v}|\overline{v}_s)^{n_s-1} \geq (v - \overline{v}_w)\, F_s(v|\overline{v}_s)^{n_s-1} > \left(v - r^{SPA}\right) F_s(v|\overline{v}_s)^{n_s-1}$$

when his type is $v \in \left[r^{SPA}, \widehat{v}\right]$. In the SPA, a bid of $\overline{v}_w$ does not even meet the reserve price. In fact, the expression on the far right hand side is an upper bound on expected payoff to a type v bidder in equilibrium in the SPA. To see this, recall that he wins with probability $F_s(v|\overline{v}_s)^{n_s-1}$ in equilibrium but that he must pay at least r^{SPA} when he wins. Thus, types below $\widehat{v}$ prefer the FPA.

For types above $\widehat{v}$, remember that Myerson (1981) has shown that expected utility in any auction can be calculated by integrating winning probabilities. Letting $q_s^{FPA}(x)$ denote the winning probability of type x in the FPA, this means that expected utility to a strong bidder with type $v \geq \widehat{v}$ in the FPA with reserve $r^{FPA} < \overline{v}_w$ is

$$\begin{aligned} U_s^{FPA}(v) &= \int_{r^{FPA}}^{v} q_s^{FPA}(x)dx \\ &= U_s^{FPA}(\widehat{v}) + \int_{\widehat{v}}^{v} q_s^{FPA}(x)dx. \end{aligned}$$

It has just been established that $U_s^{FPA}(\widehat{v})$ is higher than its counterpart in the SPA.

Likewise, the strong bidder wins at least as often in the FPA as in the SPA when his type is above $\widehat{v}$. Hence, $q_s^{FPA}(x)$ is no smaller than it would have been in the SPA. In sum, $U_s^{FPA}(v)$ is therefore higher than in the SPA. It now follows that all types above r^{FPA} strictly prefers the FPA. Hence, the FPA is strictly preferable to the SPA ex ante to all bidders as well as to the seller. ■

Proof of Proposition 4. The proof relies on extending the result in Lemma 2 and then applying the general arguments in the proof of Proposition 3. The proof of Proposition 3 made use of the assumption that $g(v) = 1$ only in order to invoke Lemma 2. If the result in Lemma 2 holds for some other $g(v)$, then Proposition 3 applies to that environment as well.

Lemma 2 proved that $k(v|r) < t(v)$ for all $v \in [r, \overline{v}_w]$ in the uniform model when $r \in [0, \overline{v}_w)$. This was a consequence of two properties: (1) $t'(v) = 1 < k'(v|r)$ and (2) $k(\overline{v}_w|r) < t(\overline{v}_w)$ when $\overline{v}_s > c(n_s, n_w)\overline{v}_w$. In the following it will be shown that $k(v|r) < t(v)$ for all $v \in [r, \overline{v}_w]$ still holds on the "relevant domain" when (2) is satisfied because both of the properties just identified move in the right direction. The "relevant domain" here is the set of reserve prices for which $J_w(r|\overline{v}_w) > 0$. The reason is that, by (1), the optimal reserve price in the SPA, should it be below $\overline{v}_w$, must satisfy $J_w(r|\overline{v}_w) > 0$. Thus, it is sufficient to prove that $\Delta(r|\overline{v}_s)$ is strictly positive and decreasing on this set of reserve prices in order to conclude that the FPA is weakly more profitable and have a weakly lower reserve price.

Step 1: Comparing the slopes of $k(v)$ and $t(v)$. Fix a reserve price, r, for which $J_w(r|\overline{v}_w) > 0$. Suppressing $\overline{v}_w$ and $\overline{v}_s$, recall that $t(v)$ is defined by $J_s(t(v)) = J_w(v)$. It follows that $t'(v) = \frac{J_w'(v)}{J_s'(t(v))}$. Thus, $t'(v) \leq 1$ if $J_w'(v) \leq J_s'(t(v))$. Note that

$$\begin{aligned} J_i'(x) &= 2 + \frac{g'(x)}{g(x)}\frac{G(\overline{v}_i) - G(x)}{g(x)} \\ &= 2 + \frac{g'(x)}{g(x)}x - \frac{g'(x)}{g(x)}J_i(x). \end{aligned}$$

Recall that $t(v) > v$ and that by definition $J_s(t(v)) = J_w(v) > 0$ for all $v \geq r$. Since $g(x)$ is log-concave, it now follows that

$$\frac{g'(t(v))}{g(t(v))}J_s(t(v)) \leq \frac{g'(v)}{g(v)}J_w(v) \text{ for all } v \geq r.$$

Likewise, (2) implies that

$$\frac{g'(v)}{g(v)}v \geq \frac{g'(t(v))}{g(t(v))}t(v).$$

It follows that $J_w'(v) \leq J_s'(t(v))$. In other words, $t'(v) \leq 1$ for all $v \geq r$ when r is such that $J_w(r) > 0$. At the same, the argument in Lemma 2 that $k'(v|r) > 1$ holds for any $g(v)$. Thus, $t'(v) \leq 1 < k'(v|r)$.

Step 2: Comparing the end-points of $k(v)$ and $t(v)$. Given the assumption that $\overline{v}_s > c(n_s, n_w)\overline{v}_w$, the proof of Lemma 2 establishes that $k(\overline{v}_w|r)$ is lower than $t(\overline{v}_w)$ when $g(v) = 1$. Since log-concavity of $g(v)$ and (2) together imply that $g(v)$ is (weakly) increasing, it follows that $F_i(v|\overline{v}_i)$ dominates the uniform distribution in terms of the likelihood-ratio and therefore in terms of the reverse hazard rate. Hubbard and Kirkegaard (2019, Proposition 3) prove that $k(\overline{v}_w|r)$ decreases when distributions change in this manner. The statement of their result assumes that $n_w \geq 2$, but they do not assume that F_s dominates F_w in terms of the reverse hazard rate. The role of the $n_w \geq 2$ assumption is to ensure that a weak bidder bids r in the FPA only if his type is r (rather than for a mass of higher types). This gives an initial condition that is crucial to the proof. In the current setting, however, F_s does in fact dominate F_w in terms of the reverse hazard rate. This implies that even if there is only a single weak bidder, he bids r only if his type is r. The reason is that a weak bidder bids more aggressively than the strong bidders and competition among the two or more strong bidders ensure that they bid r only if their type is r. Thus, the initial conditions are the same as in the proof of Hubbard and Kirkegaard (2019, Proposition 3), even when $n_w \geq 1$. Thus, $k(\overline{v}_w|r)$ is lower than it would be in the uniform model. At the same time, $t(\overline{v}_w)$ is higher under the condition in (2) than when $g(v) = 1$, as explained in the text preceding the proposition. This means that it is still the case that $k(\overline{v}_w|r) < t(\overline{v}_w)$.

Step 3: Invoking Proposition 3. Steps 1 and 2 establish that $k(v|r) < t(v)$ for all $v \in [r, \overline{v}_w]$ when $r < \overline{v}_w$ and $J_w(r|\overline{v}_w) > 0$. It follows from the arguments in the proof of Proposition 3 that $\Delta(r|\overline{v}_s)$ is strictly positive and strictly decreasing on this set of reserve prices. Proposition 4 then follows. ■

Excerpt from Kirkegaard (2021) – NOT FOR PUBLICATION

Describing the problem: The supports of F_s and F_w are $[\underline{v}_s, \overline{v}_s]$ and $[\underline{v}_w, \overline{v}_w]$, respectively. Assume that $\overline{v}_s > \overline{v}_w > \underline{v}_s \geq \underline{v}_w$. It is also assumed that F_s dominates F_w in terms of the reverse hazard rate, $F_w \leq_{rh} F_s$, or

$$\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)} \text{ for all } v \in (\underline{v}_s, \overline{v}_w]. \tag{3}$$

In other words, $\frac{F_s(v)}{F_w(v)}$ is non-decreasing on $(\underline{v}_s, \overline{v}_w]$.

I first outline two formulations of the problem. To begin, let $\varphi_i(b)$ denote bidder i's inverse bidding strategy, $b \in \left[r, \overline{b}_i\right]$, $i = s, w$. It follows from Lebrun's (2006) equilibrium characterization that

$$\widehat{v} = \min\left\{\overline{v}_s, \frac{n_s}{n_s - 1}\overline{v}_w - \frac{1}{n_s - 1}\overline{b}_w\right\}. \tag{4}$$

On the range of bids where both groups of bidders are active, $\left[r, \overline{b}_w\right]$, $\varphi_w(b)$ and $\varphi_s(b)$ solve the system of differential equations described by

$$\frac{d}{db}\ln F_i(\varphi_i(b)) = \frac{1}{n_s + n_w - 1}\left[\frac{n_j}{\varphi_j(b) - b} - \frac{n_j - 1}{\varphi_i(b) - b}\right], \tag{5}$$

$i, j = s, w$, $i \neq j$, with boundary conditions $\varphi_w(\overline{b}_w) = \overline{v}_w$ and $\varphi_s(\overline{b}_w) = \widehat{v}$. Note that if $\widehat{v} < \overline{v}_s$, then $\varphi_w'(\overline{b}_w) = 0$, by (4). Lebrun (2006) proves that $\varphi_i'(b) > 0$ for all interior bids, however.

Second, consider the formulation of the problem in terms $b_w(v)$ and $k(v)$. If his type is v, a weak bidder's problem can be thought of as deciding which type, x, to mimic. His problem is thus to maximize

$$(v - b_w(x))F_s(k(x))^{n_s}F_w(x)^{n_w - 1}.$$

Similarly, a strong bidder with type $k(v)$ who bids in the common range maximizes

$$(k(v) - b_w(x))\, F_s(k(x))^{n_s - 1}F_w(x)^{n_w}.$$

By definition of equilibrium, bidders' payoffs are maximized when $x = v$. When $v \in (r, \overline{v}_w)$, the first order conditions yield the system of differential equations

$$\begin{aligned} k'(v) &= \frac{F_s(k(v))}{f_s(k(v))}\frac{f_w(v)}{F_w(v)}T(k(v), b_w(v), v) \\ b_w'(v) &= \frac{f_w(v)}{F_w(v)}\left(k(v) - b_w(v)\right)\left[(n_s - 1)\, T(k(v), b_w(v), v) + n_w\right], \end{aligned} \tag{6}$$

where

$$T(k, b_w, v) = \frac{n_w \frac{k-b_w}{v-b_w} - (n_w - 1)}{n_s - (n_s - 1)\frac{k-b_w}{v-b_w}}$$

To compare this formulation of the problem with the previous one, the boundary conditions are that $k(\overline{v}_w) = \widehat{v}$ and $b_w(\overline{v}_w) = \overline{b}_w$.[14] Note that $T(k, b_w, v) \gtreqless 1$ if and only if $k \gtreqless v$. Likewise, holding b_w and v fixed, $T(k, b_w, v)$ is strictly increasing in k. It also holds that $\frac{\partial T(k,b_w,v)}{\partial b_w} \gtreqless 0$ if and only if $k \gtreqless v$. These properties will be used repeatedly. ▲

Lemma (Kirkegaard (2021, Lemma 1)): Assume $r \in [\underline{v}_s, \overline{v}_w)$. Then, $k(v) > v$ for all $v \in (r, \overline{v}_w]$.

Proof. Given these preliminaries it is now possible to prove Lemma 1. Recall that $k(\overline{v}_w) > \overline{v}_w$. To illustrate the proof idea, assume first that the inequality in (3) is strict. Assume there exists some $v_0 \in (r, \overline{v}_w]$ for which $k(v_0) = v_0$. Since $T = 1$ at such a point,

$$k'(v_0) = \frac{F_s(v_0)}{f_s(v_0)}\frac{f_w(v_0)}{F_w(v_0)} < 1.$$

Thus, increasing v beyond v_0 leads to the conclusion that $k(v) \leq v$. However, this contradicts the equilibrium feature that $k(\overline{v}_w) > \overline{v}_w$. The idea is the same when the inequality in (3) is weak. More formally, assume once again that there exists some $v_0 \in (r, \overline{v}_w)$ for which $k(v_0) = v_0$. Based on this "initial condition", the next step is to obtain the solution to the system of differential equations as v increases beyond v_0 (the solution to this initial value problem is unique given the differentiability assumptions imposed on the primitives). To begin, the guess is made that the solution satisfies

[14] In equilibrium, $k'(v) > 0$ and $b_w'(v) > 0$. Note, however, that if $\widehat{v} < \overline{v}_s$ then $T(k(v), b_w(v), v)$ goes to infinity as v approaches $\overline{v}_s$, by (4).

$k(v) \leq v$ for all $v \geq v_0$. Then, $T \leq 1$, and it follows that

$$\frac{d}{dv} \ln F_s(k(v)) = \frac{f_s(k(v))}{F_s(k(v))} k'(v) \leq \frac{f_w(v)}{F_w(v)} = \frac{d}{dv} \ln F_w(v),$$

or

$$\frac{d}{dv} \ln \frac{F_s(k(v))}{F_w(v)} \leq 0$$

independently of $b_w(v)$. By Gronwall's inequality, the actual solution is then bounded above by the solution that would be obtained if the above inequality had been replaced by an equality, in which case $\ln \frac{F_s(k(v))}{F_w(v)}$ would be constant. Hence, using the initial condition that $k(v_0) = v_0$,

$$\ln \frac{F_s(k(v))}{F_w(v)} \leq \ln \frac{F_s(v_0)}{F_w(v_0)}. \tag{7}$$

However, since $v \geq v_0$ reverse hazard rate dominance implies that

$$\frac{F_s(v)}{F_w(v)} \geq \frac{F_s(v_0)}{F_w(v_0)},$$

and so (7) necessitates that $k(v) \leq v$. Thus, the initial guess that $k(v) \leq v$ for all $v \geq v_0$ when $k(v_0) = v_0$ is verified. The proof is then completed in the same manner as before. In particular, the implication that $k(\overline{v}_w) \leq \overline{v}_w$ violates the equilibrium property that $k(\overline{v}_w) > \overline{v}_w$. Hence, there can be no $v_0 \in (r, \overline{v}_w)$ for which $k(v_0) = v_0$. By continuity, it then follows that $k(v) > v$ for all $v \in (r, \overline{v}_w]$. ■

Proposition (Kirkegaard (2021, Proposition 1)): Assume $n_s \geq 2$, $n_w \geq 1$. If $\overline{v}_w > r' > r \geq \underline{v}_s$ then

$$k(v|r', n_s, n_w) < k(v|r, n_s, n_w) \text{ for all } v \in [r', \overline{v}_w).$$

Proof. I first establish that the initial conditions are that $b_w(r) = r$ and $k(r) = r$. Lebrun (2006) shows that in general $\varphi_i(r) = r$ for all but at most one bidder i; see his conditions (2′) and (2″) along with his discussion on page 143. Stated differently, it is possible that $\varphi_i(r) > r$ for exactly one bidder, such that bidder i has a mass of types that bids r. However, since strategies within any given group is symmetric and $n_s \geq 2$, no strong bidder can bid r for a mass of types. The same holds for weak bidders if $n_w \geq 2$. This leaves the case where $n_w = 1$. Compared to Lebrun (2006), however, here it is assumed that reverse hazard rate dominance applies. By Lemma 1, the weak bidder is more aggressive than the strong bidders, for comparable types. Thus, the weak bidder cannot, in equilibrium, be bidding r for a mass of types. In short, it must hold that $\varphi_i(r) = r$ for all bidders in the current model. Equivalently, the initial conditions to the system in (6) are that $k(r) = r$ and $b_w(r) = r$.

Let $\widehat{v}$ denote the strong bidders' cut-off type and $\overline{b}_w$ the weak bidders' maximum bid when the reserve price is r. Let $\widehat{v}'$ and $\overline{b}_w'$ denote their counterparts when the reserve price increases to r'. Note first that if $\overline{b}_w = \overline{b}_w'$ then $\widehat{v} = \widehat{v}'$, by (4). The system of differential equations are then characterized by the same boundary conditions regardless of whether the reserve price is r or r'. Thus, the system is the same on $b \in (r', \overline{b}_w]$ in either case. Given the differentiability assumptions imposed on the primitives, the unique solution to the two problems must then coincide on $b \in (r', \overline{b}_w]$. Hence, in the limit, $b_w(r'|r') = b_w(r'|r)$. However, the initial conditions when the reserve price is r' requires $b_w(r'|r') = r'$, whereas equilibrium bidding when the reserve price is $r < r'$ satisfies $b_w(r'|r) < r'$. This contradicts the previous conclusion that $b_w(r'|r') = b_w(r'|r)$. Thus, in equilibrium, $\overline{b}_w \neq \overline{b}_w'$.

Consider next the possibility that $\overline{b}_w > \overline{b}_w'$, implying that $\widehat{v}' \geq \widehat{v}$, by (4). Assume first that $\widehat{v}' > \widehat{v}$. Hence, for v close to $\overline{v}_w$, $k(v|r')$ is strictly above $k(v|r)$ while $b_w(v|r')$ is strictly below $b_w(v|r)$, or $k(\overline{v}_w|r') = \widehat{v}' > \widehat{v} = k(\overline{v}_w|r)$ and $b_w(\overline{v}_w|r') = \overline{b}_w' < \overline{b}_w = b_w(\overline{v}_w|r)$. Reducing v from $\overline{v}_w$, find the nearest value, v', (if one exists) where one of the new endogenous functions crosses its old counterpart. The argument in the previous paragraph rules out that $k(v'|r') = k(v'|r)$ and $b_w(v'|r') = b_w(v'|r)$ at the same time. There are two remaining cases. Assume $b_w(v'|r') = b_w(v'|r)$ but $k(v'|r') > k(v'|r)$. Then, from (6), $b_w'(v'|r') > b_w'(v'|r)$. This contradicts that $b_w(v|r') < b_w(v|r)$ for $v > v'$. Assume instead that $k(v'|r') = k(v'|r)$ but $b_w(v'|r') < b_w(v'|r)$. Then, again from (6), $k'(v'|r') < k'(v'|r)$ if $k(v'|r') = k(v'|r) > v'$. However, this contradicts that $k(v|r') > k(v|r)$ for $v > v'$.

Next, assume that $\overline{b}_w > \overline{b}_w'$ but that $\widehat{v}' = \widehat{v}$. This necessitates $\widehat{v}' = \widehat{v} = \overline{v}_s$. It can now be seen that $k(v|r)$ is steeper than $k(v|r')$ near $\overline{v}_w$. Hence, $k(v|r') > k(v|r)$ for v close to, but strictly below, $\overline{v}_w$. By continuity, it is also the case that $b_w(v|r') < b_w(v|r)$ in such a neighborhood. The previous arguments can then be repeated to obtain a contradiction.

Hence, it has now been shown that $\overline{b}_w < \overline{b}_w'$, thereby implying that $\widehat{v}' \leq \widehat{v}$. Stated differently, $b_w(\overline{v}_w|r) < b_w(\overline{v}_w|r')$ and $k(\overline{v}_w|r) \geq k(\overline{v}_w|r')$. Moreover, either $k(\overline{v}_w|r) > k(\overline{v}_w|r')$ or $k(v|r)$ is flatter than $k(v|r')$ near $\overline{v}_w$. In either case, $b_w(v|r) < b_w(v|r')$ and $k(v|r) > k(v|r')$ when v is close to $\overline{v}_w$. Arguments like those above can then be used to prove that these inequalities are unchanged as v is reduced from $\overline{v}_w$ to r'. ■

Proposition (Kirkegaard (2021, Proposition 3)): Assume $n_s \geq 2$, $n_w \geq 1$, and $r \in [\underline{v}_s, \overline{v}_w)$. Then $k(v|r, n_s, n_w) \to v$ for all $v \in (r, \overline{v}_w]$ as $n_s \to \infty$ or $n_w \to \infty$.

Proof of Proposition 3. Lemma 1 establishes the lower bound that $k(v) > v$ for all $v \in (r, \overline{v}_w]$. An upper bound on $k(v)$ is derived next. The proof then concludes by showing that the upper bound converges to v as the number of bidders goes to infinity.

Using (5) and the condition that $\varphi_w'(b) \geq 0$ yield the conclusion that

$$\frac{n_s}{k(v) - b_w(v)} - \frac{n_s - 1}{v - b_w(v)} \geq 0$$

or

$$k(v) \leq \frac{n_s}{n_s - 1} v - \frac{1}{n_s - 1} b_w(v) \tag{8}$$

for all $v \in (r, \overline{v}_w]$. Since $b_w(v)$ is bounded above by v, the last term in (8) goes to zero as $n_s \to \infty$. Since the first term converges to v, it now follows that $k(v) \to v$ as $n_s \to \infty$.

Next, consider changes in n_w instead. In equilibrium, $b_w(v) \leq v$. At the same time, it follows from Myerson (1981) that for any $v \in (r, \overline{v}_w]$,

$$(v - b_w(v))\, F_w(v)^{n_w - 1} F_s(k(v))^{n_s} = \int_r^v F_w(x)^{n_w-1} F_s(k(x))^{n_s} dx$$

or

$$\begin{aligned} b_w(v) &= v - \int_r^v \left(\frac{F_w(x)}{F_w(v)}\right)^{n_w-1} \left(\frac{F_s(k(x))}{F_s(k(v))}\right)^{n_s} dx \\ &\geq v - \int_r^v \left(\frac{F_w(x)}{F_w(v)}\right)^{n_w-1} dx \to v \text{ as } n_w \to \infty. \end{aligned}$$

Thus, $b_w(v) \to v$ as $n_w \to \infty$. Once again, (8) now implies that $k(v) \to v$ as $n_w \to \infty$. ■

Excerpt from Hubbard and Kirkegaard (2019) – NOT FOR PUBLICATION

Hubbard and Kirkegaard (2019, Proposition 3)

Remark: The following describes Hubbard and Kirkegaard's (2019) Proposition 3 in its original form but with notation that fits the current paper. This includes the assumption that $n_w \geq 2$. The proof of Proposition 4 in the current paper explains why this can be relaxed to $n_w \geq 1$ in the current setting. To apply Hubbard and Kirkegaard (2019, Proposition 3), think of F_i as being the uniform distribution and H_i as being the new distribution derived from a density function that satisfies (2).

Fix the size of the two groups, $n_s, n_w \geq 2$ and the supports $[\underline{v}_s, \overline{v}_s]$ and $[\underline{v}_w, \overline{v}_w]$, respectively. Let $\underline{v} = \max\{\underline{v}_s, \underline{v}_w\}$. We let the pair of distributions change from (F_s, F_w) to (H_s, H_w). Recall that H_i (strictly) dominates F_i in terms of the reverse hazard rate if

$$\frac{h_i(v)}{H_i(v)} > \frac{f_i(v)}{F_i(v)} \text{ for all } v \in (\underline{v}_i, \overline{v}_i].$$

Borrowing from Lebrun (1998), we will write $H_i \succ F_i$ if the above holds. Here, we will assume that H_i either strictly dominates F_i in terms of the reverse hazard rate or is identical to F_i. Borrowing from Lebrun (1998) once again, this will be denoted $H_i \succeq F_i$.

Let $(\widehat{v}^F, \overline{b}_w^F)$ and $(\widehat{v}^H, \overline{b}_w^H)$ denote the equilibrium values of $(\widehat{v}, \overline{b}_w)$ when the distributions are (F_s, F_w) and (H_s, H_w), respectively. The characterization in Hubbard and Kirkegaard (2019, Proposition 1) implies that $\widehat{v}$ is non-increasing with $\overline{b}_w$.

Proposition (Hubbard and Kirkegaard (2019, Proposition 3)): Assume $H_i \succeq F_i$, $i = s, w$ and $n_s, n_w \geq 2$. Assume there is a binding reserve price in place, with $r \in (\underline{v}, \min\{\overline{v}_s, \overline{v}_w\})$. Then, $\overline{b}_w^H \geq \overline{b}_w^F$ and thus $\widehat{v}^H \leq \widehat{v}^F$. Consequently, if bid-separation occurs under (F_s, F_w) it also occurs under (H_s, H_w); i.e., when bidders become stronger.

Proof. Let bidder i's inverse bidding strategy be denoted $\varphi_i^F(b)$ and $\varphi_i^H(b)$ in the two scenarios where distributions are (F_s, F_w) and (H_s, H_w), respectively (recall that equilibrium is unique). The case where $H_s = F_s$ and $H_w = F_w$ is uninteresting. Thus, assume in the remainder that $H_s \succ F_s$ and/or $H_w \succ F_w$. The original system of differential equation can be written as

$$\varphi_i'(b) = \frac{1}{n_s + n_w - 1} \frac{F_i\left(\varphi_i(b)\right)}{f_i\left(\varphi_i(b)\right)} \left[\frac{n_j}{\varphi_j(b) - b} - \frac{n_j - 1}{\varphi_i(b) - b}\right] \tag{9}$$

where $j \neq i$, $i = s, w$.

Assume by contradiction that $\overline{b}_w^H < \overline{b}_w^F$, which by Hubbard and Kirkegaard (2019, Proposition 1) [or (10) in the next section] implies $\widehat{v}^H \geq \widehat{v}^F$. Hence, $\varphi_i^H(\overline{b}_w^H) > \varphi_i^F(\overline{b}_w^H)$, $i = s, w$. Now move leftwards (reducing b) from $\overline{b}_w^H$. Let $b' > r$ denote the first (i.e. highest) bid for which $\varphi_i^H(b') = \varphi_i^F(b')$ for some (or both) i, if it exists. If it exists, there are two possibilities. One possibility is that the crossing occurs at the same place for both $i = s$ and $i = w$, i.e. $\varphi_s^H(b') = \varphi_s^F(b')$ and $\varphi_w^H(b') = \varphi_w^F(b')$. The bracketed term in (9) is then the same for both scenarios. However, since $H_i \succ F_i$ for some i, it follows that $\varphi_i^{H\prime}(b') < \varphi_i^{F\prime}(b')$. However, this contradicts the fact that $\varphi_i^H(b) > \varphi_i^F(b)$ to the right of b'. The other possibility is that $\varphi_i^H(b') = \varphi_i^F(b')$ but $\varphi_j^H(b') > \varphi_j^F(b')$, $j \neq i$. The conclusion is again that $\varphi_i^{H\prime}(b') < \varphi_i^{F\prime}(b')$, because the bracketed term in (9) is smaller (and the first term is no larger) when distributions are (H_s, H_w) compared to (F_s, F_w). The same contradiction is thus achieved. It now follows that $\varphi_i^H(b) > \varphi_i^F(b)$ for all $b \in (r, \overline{b}_w^H]$, $i = s, w$. Assuming that $H_s \succ F_s$ (the proof is similar if $H_w \succ F_w$ instead) it follows from the bidders' first order conditions that

$$\begin{aligned}
\frac{d}{db} \ln F_w(\varphi_w^F(b))^{n_w} F_s(\varphi_s^F(b))^{n_s - 1} &= \frac{1}{\varphi_s^F(b) - b} \\
&> \frac{1}{\varphi_s^H(b) - b} \\
&= \frac{d}{db} \ln H_w(\varphi_w^H(b))^{n_w} H_s(\varphi_s^H(b))^{n_s - 1},
\end{aligned}$$

or

$$\frac{d}{db}\left[\ln\left(\frac{F_w(\varphi_w^F(b))}{F_w(\varphi_w^F(\overline{b}_w^H))}\right)^{n_w}\left(\frac{F_s(\varphi_s^F(b))}{F_s(\varphi_s^F(\overline{b}_w^H))}\right)^{n_s-1}\right] > \frac{d}{db}\left[\ln\left(\frac{H_w(\varphi_w^H(b))}{H_w(\varphi_w^H(\overline{b}_w^H))}\right)^{n_w}\left(\frac{H_s(\varphi_s^H(b))}{H_s(\varphi_s^H(\overline{b}_w^H))}\right)^{n_s-1}\right]$$

for all $b \in (r, \overline{b}_w^H]$. Evaluated at $b = \overline{b}_w^H$, the bracketed term on either side of the inequality are both zero. Since $r > \underline{v}$, the bracketed term on the left converges to a finite value as $b \to r$. Moreover, since the bracketed term on the left is steeper in b than the bracketed term on the right, the latter must also converge to a finite value, with

$$\ln\left(\frac{F_w(\varphi_w^F(r))}{F_w(\varphi_w^F(\bar{b}_w^H))}\right)^{n_w}\left(\frac{F_s(\varphi_s^F(r))}{F_s(\varphi_s^F(\bar{b}_w^H))}\right)^{n_s-1} < \ln\left(\frac{H_w(\varphi_w^H(r))}{H_w(\varphi_w^H(\bar{b}_w^H))}\right)^{n_w}\left(\frac{H_s(\varphi_s^H(r))}{H_s(\varphi_s^H(\bar{b}_w^H))}\right)^{n_s-1}$$

Since $(\varphi_s^F, \varphi_w^F)$ are equilibrium strategies, it must hold that $\varphi_i^F(r) = r$, $i = s, w$. If $(\varphi_s^H, \varphi_w^H)$ are equilibrium strategies as well, then it also holds that $\varphi_i^H(r) = r$, $i = s, w$, and we have

$$\left(\frac{F_w(r)}{F_w(\varphi_w^F(\bar{b}_w^H))}\right)^{n_w}\left(\frac{F_s(r)}{F_s(\varphi_s^F(\bar{b}_w^H))}\right)^{n_s-1} < \left(\frac{H_w(r)}{H_w(\varphi_w^G(\bar{b}_w^H))}\right)^{n_w}\left(\frac{H_s(r)}{H_s(\varphi_s^H(\bar{b}_w^H))}\right)^{n_s-1}$$

or

$$\left(\frac{H_w(\varphi_w^H(\bar{b}_w^H))}{F_w(\varphi_w^F(\bar{b}_w^H))}\right)^{n_w}\left(\frac{H_s(\varphi_s^H(\bar{b}_w^H))}{F_s(\varphi_s^F(\bar{b}_w^H))}\right)^{n_s-1} < \left(\frac{H_w(r)}{F_w(r)}\right)^{n_w}\left(\frac{H_s(r)}{F_s(r)}\right)^{n_s-1}$$

which, since $\varphi_i^H(\bar{b}_w^H) > \varphi_i^F(\bar{b}_w^H)$, implies

$$\left(\frac{H_w(\varphi_w^H(\bar{b}_w^H))}{F_w(\varphi_w^H(\bar{b}_w^H))}\right)^{n_w}\left(\frac{H_s(\varphi_s^H(\bar{b}_w^H))}{F_s(\varphi_s^H(\bar{b}_w^H))}\right)^{n_s-1} < \left(\frac{H_w(r)}{F_w(r)}\right)^{n_w}\left(\frac{H_s(r)}{F_s(r)}\right)^{n_s-1}$$

However, the assumption that $H_i \succ F_i$ $(H_i = F_i)$ is equivalent to $\frac{d}{dv}\frac{H(v)}{F(v)} > 0$ $(\frac{d}{dv}\frac{H(v)}{F(v)} = 0)$. Consequently, the above inequality must be violated. In other words, $(\varphi_s^H, \varphi_w^H)$ cannot form an equilibrium. ■

Hubbard and Kirkegaard (2019, Proposition 5)

Preliminaries: There are $n_s \geq 2$ strong bidders, with $F_s(v) = \frac{v}{\overline{v}_s}$, $v \in [0, \overline{v}_s]$. There are $n_w \geq 1$ weak bidders, with $F_w(v) = \frac{v}{\overline{v}_w}$, $v \in [0, \overline{v}_w]$, and $\overline{v}_s > \overline{v}_w > 0$. Let $\bar{b}_w$ and $\bar{b}_s$ denote the bid of a weak bidder with type $\overline{v}_w$ and a strong bidder with type $\overline{v}_s$, respectively. For ease of notation in formulating the result, let $n = n_s + n_w - 1$ denote the number of rivals faced by any bidder. Finally, define $\kappa(n_s, n_w)$ and $\tau(n_s, n_w)$, respectively, as

$$\kappa(n_s, n_w) = \frac{(n_s + 1)\, n - \sqrt{(n_s + 1)^2\, n^2 - 4 n_w n_s n}}{2 n_w},$$
$$\tau(n_s, n_w) = \frac{n_s - \kappa(n_s, n_w)}{n_s - 1},$$

and note that $\kappa(n_s, n_w) \in (0, 1)$ while $\tau(n_s, n_w) > 1$. Of course, both functions are independent of $\overline{v}_s$ and $\overline{v}_w$. It can be shown that $\kappa(n_s, n_w)$ is strictly increasing in both its arguments and that $\tau(n_s, n_w)$ is strictly decreasing in both its arguments.

Proposition (Hubbard and Kirkegaard (2019, Proposition 5)): Assume $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = s, w$, with $\overline{v}_s > \overline{v}_w > 0$. Assume $n_s \geq 2$, $n_w \geq 1$. Equilibrium properties depend on the relative difference between supports, $\frac{\overline{v}_s}{\overline{v}_w}$:

1. If $\frac{\overline{v}_s}{\overline{v}_w} \leq \tau(n_s, n_w)$ (the supports do not differ too much), then both kinds of bidders share the same maximum bid, $\overline{b}_w = \overline{b}_s$ and $\widehat{v} = \overline{v}_s$, with

$$\overline{b}_w = \frac{n}{\overline{v}_s n_w + \overline{v}_w n_s} \overline{v}_s \overline{v}_w.$$

2. If $\frac{\overline{v}_s}{\overline{v}_w} > \tau(n_s, n_w)$ (the supports differ considerably), then bid-separation occurs, $\overline{b}_w < \overline{b}_s$ and $\widehat{v} < \overline{v}_s$, with

$$\overline{b}_w = \kappa(n_s, n_w) \overline{v}_w$$

and

$$\widehat{v} = \tau(n_s, n_w) \overline{v}_w.$$

Moreover, the equilibrium is continuous in all the parameters, n_s, n_w, $\overline{v}_s$ and $\overline{v}_w$.

Proof. It follows from Lebrun (2006) that for any $\overline{b}_w \in (0, \overline{v}_w)$ candidate, $\widehat{v}$ is uniquely determined by

$$\widehat{v} = \min \left\{ \overline{v}_s, \frac{n_s}{n_s - 1} \overline{v}_w - \frac{1}{n_s - 1} \overline{b}_w \right\}. \tag{10}$$

Hubbard and Kirkegaard (2019) explain how (10) is derived. The relationship in (10) characterizes a necessary condition on any candidate $(\widehat{v}, \overline{b}_w)$ pair. The next step is to use mechanism design arguments to derive a second necessary condition. The final step combines these two conditions to establish the proposition.

As in any mechanism design argument, the equilibrium allocation plays an important role. Thus, let $q_i(v)$ denote the probability that a bidder in group i, $i = s, w$, wins the auction if his type is v. Letting $EU_i(v)$ denote such a bidder's expected utility, Myerson (1981) has shown that

$$EU_i(v) = EU_i(0) + \int_0^v q_i(x)dx.$$

It is easily seen that $EU_i(v0) = 0$. Consider now the highest types, $\overline{v}_s$ and $\overline{v}_w$, respectively. First, observe that

$$\begin{aligned} EU_s(\overline{v}_s) &= EU_s(\widehat{v}) + \int_{\widehat{v}}^{\overline{v}_s} q_i(x)dx \\ &= \left(\widehat{v} - \overline{b}_w\right)\left(\frac{\widehat{v}}{\overline{v}_s}\right)^{n_s-1} + \int_{\widehat{v}}^{\overline{v}_s}\left(\frac{x}{\overline{v}_s}\right)^{n_s-1} dx, \end{aligned}$$

since type $x \geq \widehat{v}$ outbids all weak bidders with probability one and thus wins if all rival bidders in group i have types that are below x. Conveniently, this expression does not require any knowledge of $q_s(x)$ for $x < \widehat{v}$. Integrating now yields the conclusion that

$$\int_0^{\overline{v}_s} q_s(x)dx = \left(\widehat{v} - \overline{b}_w\right)\left(\frac{\widehat{v}}{\overline{v}_s}\right)^{n_s-1} + \frac{1}{n_s}\frac{\overline{v}_s^{n_s} - \widehat{v}^{n_s}}{\overline{v}_s^{n_s}}. \tag{11}$$

Similarly, since

$$EU_w(\overline{v}_w) = \left(\overline{v}_w - \overline{b}_w\right)\left(\frac{\widehat{v}}{\overline{v}_s}\right)^{n_s},$$

it follows that

$$\int_0^{\overline{v}_w} q_s(x)dx = \left(\overline{v}_w - \overline{b}_w\right)\left(\frac{\widehat{v}}{\overline{v}_s}\right)^{n_s}. \tag{12}$$

The ex ante probability that any given bidder wins the auction takes a particularly useful form when distributions are uniform, since

$$\int_0^{\overline{v}_i} q_i(x)f_i(x)dx = \frac{1}{\overline{v}_i}\int_0^{\overline{v}_i} q_i(x)dx.$$

Since the auction has no reserve price, the item will be sold for sure. In other words, the ex ante winning probabilities must aggregate to one, or

$$n_s \frac{1}{\overline{v}_s} \int_0^{\overline{v}_s} q_s(x)dx + n_w \frac{1}{\overline{v}_w} \int_0^{\overline{v}_w} q_w(x)dx = 1. \tag{13}$$

Combining (11) and (12) with (13) yields the necessary condition that

$$\overline{b}_w = \frac{n}{\overline{b}_w n_s + n_w \widehat{v}} \overline{v}_w \widehat{v} \tag{14}$$

for $\widehat{v} \in [0, \overline{v}_s]$, or, stated differently,

$$\widehat{v} = \frac{\overline{v}_w n_s \overline{b}_w}{n_w \left(\overline{v}_w - \overline{b}_w\right) + \overline{v}_w \left(n_s - 1\right)} \tag{15}$$

with the restriction that $\overline{b}_w$ is such that $\widehat{v} \in [0, \overline{v}_s]$.

In summary, any equilibrium $(\widehat{v}, \overline{b}_w)$ pair must satisfy both (15) and (10). Thus, the next step is to characterize what turns out to be the unique $(\widehat{v}, \overline{b}_w)$ pair that satisfies both conditions. First, note that the right hand side of (15) is strictly increasing in $\overline{b}_w$ and ranges from 0 to $\frac{n_s}{n_s-1}\overline{v}_w$ as $\overline{b}_w$ increases from 0 to $\overline{v}_w$. However, the term $\frac{n_s}{n_s-1}\overline{v}_w - \frac{1}{n_s-1}\overline{b}_w$ on the right hand side of (10) is strictly decreasing in $\overline{b}_w$ and ranges from $\frac{n_s}{n_s-1}\overline{v}_w$ to $\overline{v}_w$ as $\overline{b}_w$ increases from 0 to $\overline{v}_w$. Thus, the two equations (i.e. (15) and $\widehat{v} = \frac{n_s}{n_s-1}\overline{v}_w - \frac{1}{n_s-1}\overline{b}_w$) must have a unique intersection with $\overline{b}_w \in (0, \overline{v}_w)$. We first identify this intersection and then subsequently check whether it satisfies the feasibility condition that $\widehat{v} \in [0, \overline{v}_s]$. Equalizing these two equations yields a quadratic equation in $\overline{b}_w$. The larger root can be ruled out, since it yields the conclusion that $\overline{b}_w > \overline{v}_w$. The smaller root is $\overline{b}_w = \kappa(n_s, n_w)\overline{v}_w$, for which $\widehat{v} = \tau(n_s, n_w)\overline{v}_w$. This candidate satisfies the final feasibility condition that $\widehat{v} \leq \overline{v}_s$ if and only if $\tau(n_s, n_w) \leq \frac{\overline{v}_s}{\overline{v}_w}$. This proves the second part of the proposition. If $\tau(n_s, n_w) > \frac{\overline{v}_s}{\overline{v}_w}$, the condition that $\widehat{v} \leq \overline{v}_s$ instead binds. Nevertheless, (15) and (14) must be satisfied. The latter establishes the characterization in the first part of the proposition.

Continuity follows from the continuity of (15) and (10). Of course this implies that when $\tau(n_s, n_w)$ is identical to $\frac{\overline{v}_s}{\overline{v}_w}$, the equilibrium pair $(\widehat{v}, \overline{b}_w)$ in the two parts of the proposition coincide. ■