

Asymmetric Auctions with More Than Two Bidders*

Timothy P. Hubbard

Colby College

timothy.hubbard@colby.edu

René Kirkegaard

University of Guelph

rkirkega@uoguelph.ca

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Abstract

The theoretical literature on asymmetric first-price auctions has focused mainly on settings with either (1) exactly two bidders or (2) an arbitrary number of bidders with types in a common support. Here, we explore the complications that arise when moving beyond these settings. The structure of the problem changes significantly as bidders may now tender bids over different intervals, a property we term bid-separation. We present a number of comparative statics results that characterize when this is likely to occur. Bid-separation is more prevalent the more bidders there are at auction and the “stronger” bidders are on their respective supports. Stretching the supports may or may not lead to more bid-separation; a complete characterization is provided for the influential uniform-distribution model. Importantly, bid-separation invalidates existing numerical solution methods. However, we propose a method that can be used to modify these to take bid-separation into account.

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1 Introduction

The interest in asymmetric auctions dates back to Vickrey's (1961) foundational work. However, the ensuing literature has primarily focused on two somewhat special models of first-price auctions. In the first model there are exactly two bidders, where typically a "weak" bidder is facing a "strong" bidder. It is not hard to see that in equilibrium the two bidders share a common maximum bid; otherwise, one bidder could lower his bid without lowering his winning probability. In the second model, there is an arbitrary number of bidders, but the distributions of bidders' types have a common support. In this case, it can also be shown that all bidders share a common maximum bid. Examples of the first kind of model are Maskin and Riley (2000) and Kirkegaard (2012), while examples of the second kind are Lebrun (1999) and Kirkegaard (2009), among many others. Technically, these models have the helpful feature that there is a single boundary condition. This makes it easier to make inferences from the system of differential equations describing equilibrium behavior.

To highlight the limitations of such models, imagine two weak bidders are facing two strong bidders. For instance, two art students are up against two well-known billionaire art collectors in a first-price auction for an old masterpiece. The latter are unlikely to be too concerned about the presence of the former, and it is patently absurd to suggest that the two kinds of bidders would share the same maximum bid. Compared to the first kind of model mentioned above, the issue here is that once the economy grows, equilibrium behavior undergoes a qualitative change. Stated differently, replicating the economy is not without its pitfalls. Compared to the second kind of model noted above, the issue is that the type supports are likely very different for students and billionaires. In either situation, it is no longer necessarily the case that there is a single boundary condition prescribing that all bidders tender the same highest bid. We refer to this equilibrium property as bid-separation since different bidders submit bids on different (albeit not disjoint) intervals. Hence, any analysis that assumes a single boundary condition should be treated with some caution. The current paper is devoted specifically to the study of bid-separation.

The complications due to bid-separation have been recognized in the theoretical literature before. However, the only paper that tackles the problem head on appears to be Lebrun (2006), but the point of that paper is primarily to establish uniqueness of the equilibrium. On the other hand, it seems that bid-separation is less widely ac-

knowledge in the empirical literature; we return to this point momentarily. Lastly, because analytic solutions to the system of differential equations describing equilibrium behavior rarely exist, applied researchers often solve for bidding strategies numerically. However, researchers contributing to that literature have rarely dealt with the possibility of bid-separation.¹ In fact, the opposite is usually true—numerical approaches often hinge critically on a common bid support.

Nevertheless, an easy practical “fix” to the issue of bid-separation has been proposed in the literature; see e.g. Li and Riley (2007). The idea is to replace bidders’ true distributions with approximations that are defined over a larger set of types in such a way that all bidders come to have the same maximum type. Hence, positive but almost negligible probability mass is transferred to an interval of types that in the true specification has zero probability of occurring. In this new environment, bidders share a common maximum bid (since they now share a common maximum type) and so standard methods can be used to solve the problem.²

We do not dispute that this method may work well in practice. Nevertheless, conceptually we do not find it entirely satisfactory. After all, an approximation error has been deliberately introduced into the set-up. In this paper, we will show how to modify existing procedures to solve the original problem without relying on approximations at all. Conceptually, then, our approach is “cleaner” and remains true to the original problem. However, it is perhaps even more important to emphasize that the approximation approach more or less by construction distracts attention away from bid-separation. In contrast, we will argue that bid-separation is more than a technical curiosity and that it in fact deserves attention for purely economic reasons. In particular, bid-separation has “design” implications that appear to have been overlooked thus far. Thus, we maintain that from a theoretical perspective it is important to understand when bid-separation occurs.

A design implication of bid-separation exists when one attempts to compare expected revenue across auction formats. Maskin and Riley (2000) and Kirkegaard (2012) address this problem in the two-bidder case. A complication that arises is

¹Exceptions include Li and Riley (2007) and Marion (2007), both of which are discussed below. Likewise, Hubbard and Paarsch (2014, footnote 8) recognize that the canonical boundary conditions may need adjustment if the type supports differ when there are more than two bidders at auction; however, they do not consider such an extension.

²By Lebrun’s (2002) continuity result the equilibrium of the perturbed model is close to the equilibrium of the true model.

that without bid-separation, a weak bidder with a high type outbids almost all other bidders, including some that have a much higher “virtual valuation” in the terminology of Myerson (1981). Bid-separation, on the other hand, by definition means that a weak bidder with a high type no longer wins as often. From a revenue perspective, this may be an advantage. Kirkegaard (2019) pursues this idea and shows that bid-separation may present the key to unlocking revenue comparisons between first-price auctions and second-price auctions with more than two bidders.

This paper makes three related contributions. The first objective is simply to draw attention to the possibility of bid-separation. As we describe in more detail below, this seemingly simplistic objective is motivated in part by current practices in the empirical literature. Similarly, we wish to convey the point that conceptually something is perhaps lost when the “approximation approach” is used. For instance, design implications are more easily overlooked (see above). The second contribution is more substantive. Specifically, we aim to understand both qualitatively and quantitatively when bid-separation arises. To this end, the core part of the paper derives a number of comparative statics results. The third contribution is methodological in nature. Specifically, we demonstrate that there is no need to employ the approximation approach in the first place. Instead, we propose a simple two-step procedure that amends standard numerical solution methods to solve the problem.

Our paper is closely related to Lebrun (2006), but our focus is decidedly different. Lebrun (2006) considers a general set-up with an arbitrary number of groups of bidders. Thus, his model is more general than ours as we assume there are only two groups of bidders. However, his focus is on existence and uniqueness of equilibrium, not on bid-separation per se. Nevertheless, bid-separation is of course implicit in his equilibrium characterization. Stated differently, our preliminary results in Section 2 – on equilibria that involve bid-separation – are implicit in Lebrun (2006). However, the proofs are new, self-contained, and deliberately constructed to make it much more explicit what drives bid-separation. In other words, in keeping with our first objective, we pursue a formulation of the problem that makes bid-separation take centre stage. We have chosen this approach partly in the hopes of reaching a broader audience, as bid-separation is of relevance to theorists, empiricists, and researchers working on numerical methods alike. The comparative statics results that we present in Section 3 are new, as is our two-step procedure in Section 4 that circumvents the approximation approach. Indeed, we would probably not have been able to develop the two-step

procedure without the more explicit insights into bid-separation that we obtain by reexamining the problem in Section 2.

As mentioned, we assume there are precisely two groups of bidders, where bidders in the same group are *ex ante* identical. This assumption is consistent with the existing empirical literature, which almost exclusively adopts such a specification.³ Moreover, recall that Lebrun (2006) has already characterized equilibrium in the general case. We specialize that setting to sharpen the focus on bid-separation.

Our interest in bid-separation is due in part to a recent literature on policy interventions in auctions. In particular, bid preference policies has received much attention due to the prevalence of such programs in procurement auctions; see, for example, Marion (2007), Hubbard and Paarsch (2009), and Krasnokutskaya and Seim (2011). It is common in this literature to simulate counterfactual auctions with various levels of preferential treatment in order to determine its optimal level. We outline such a policy in a first-price setting and note that the level of asymmetry is effectively endogenously determined by the size of the preferential treatment. Specifically, in a standard first-price auction, a bidder's expected utility if his valuation is v and his bid is b is $(v - b)G(b)$, where $G(b)$ is the (endogenously determined) winning probability. Now assume this bidder is given preferential treatment. For instance, if he wins he has to pay only a fraction $r \in (0, 1)$ of his bid. As bidding strategies are likely to change, the winning probability for any given bid is likely to change as well. Thus, let $G_r(b)$ denote the new winning probability, given b and r . The bidder's expected utility is now $(v - rb)G_r(b)$, which is of course maximized where $(v/r - b)G_r(b)$ is maximized. Thus, from his competitors' point-of-view, giving the bidder preference is equivalent to changing the distribution of his types. It is as if the support shifts to the right. Even if all bidders start out with the same support, preferential treatment effectively destroys that property. Thus, bid-separation may become an issue in the counterfactual analysis, although this is not always acknowledged in the literature.

Sections 3 is devoted to comparative statics. As expected, bid-separation is more

³To give a few examples, Campo, Perrigne, and Vuong (2003) distinguish between joint and solo bidders, De Silva, Dunne, and Kosmopoulou (2003) classify bidders as entrant or incumbents, Flambar and Perrigne (2006) use firm location as a source of asymmetry, Marion (2007) as well as Krasnokutskaya and Seim (2011) both identify state-qualified small business enterprises and non-small businesses, while Athey, Levin, and Seira (2011) consider loggers and sawmills as inherently different. The only empirical paper we know of that treats more than two different groups of bidders is De Silva, Hubbard, and Kosmopoulou (2015). However, due to data constraints, they only deal with auctions in which combinations of two groups of bidders are present in their application.

likely to occur the more bidders that are participating. The same is true when bidders in the strong group become even stronger. However, when bidders in the weak group become stronger, bid-separation may become more or less likely, depending on how their increased strength is modeled. Thus, it is not always the case that bid-separation becomes less prevalent when the asymmetry between bidders diminishes. Heuristically, the auction tends to become more efficient when the asymmetry diminishes. Moreover, bid-separation implies that the allocation is efficient when a strong bidder has a sufficiently high type. Thus, holding the type supports fixed, it is natural to expect to see more bid-separation when the weak bidders become more similar to the strong bidders on the intersection of their respective supports. Conversely, if the support of the weak bidders' types also changes – specifically, if their highest type increases – then efficiency improves when there is less bid-separation.

The last part of Section 3 adds to the above qualitative results by presenting results of a more quantitative nature. We offer a detailed examination of a model in which type distributions are uniform over different supports. In this case it is possible to quantify precisely how large the asymmetry between bidders must be in order for bid-separation to arise. Stated differently, we derive necessary and sufficient conditions for bid-separation in this environment. For example, if there are three strong bidders and two weak bidders, with supports $[0, \bar{v}_1]$ and $[0, \bar{v}_2]$ respectively, bid-separation results as long as \bar{v}_1 is more than 8.11% percent larger than \bar{v}_2 . This percentage rapidly decreases in the number of bidders.⁴ Our interpretation is that these asymmetries are so small that one should not a priori dismiss the possibility of bid-separation in real-world applications.

Section 4 turns to the more practical problem of simulating equilibrium when closed form solutions are not available. As indicated above, we present a two-step procedure that obviates the need for Li and Riley's (2007) approximation approach. The first step consists of a novel and quick “test” for whether or not bid-separation arises in a given auction environment. Hence, only minimal work is required to determine if bid-separation will occur in equilibrium. Another advantage of the test is that it can be applied across a class of parameterized auction environments to derive “numerical” comparative statics on when, i.e. for which parameter values,

⁴The model in which bidders' types are uniformly distributed over different supports has been extremely valuable to theorists since Vickrey's (1961) groundbreaking work. Kaplan and Zamir (2012) recently generalized Vickrey's equilibrium characterization. Both papers assume there are precisely two bidders. We continue in this tradition, but allow for more bidders.

bid-separation occurs. We illustrate this by presenting an example that numerically yields quantitative results that complements the theoretical results in Section 3. In the second step of the solution procedure, we detail how to modify any off-the-shelf solution procedure for auctions without bid-separation to handle bid-separation if the first step has indicated that this must be a feature of equilibrium bidding.

2 Model and preliminaries

We begin by outlining a general model which is then later specialized. A total of n risk neutral bidders are participating in an independent private-values first-price auction. Bidder i 's value or type, v_i , is drawn from the distribution function F_i , with continuous and strictly positive density f_i and support $[\underline{v}_i, \bar{v}_i]$, $\bar{v}_i > \underline{v}_i \geq 0$. Lebrun (2006) has proven that there is an essentially unique equilibrium in undominated strategies under the mild assumption that F_i is strictly log-concave near \underline{v}_i , $i = 1, 2, \dots, n$.⁵ In this paper we simply assume that the equilibrium is unique. Bids are continuous and strictly increasing in type among those types that have a strictly positive probability of winning. Let \bar{b}_i denote the bid submitted by bidder i with type \bar{v}_i .

It is intuitive that \bar{b}_i is higher the higher \bar{v}_i is. Indeed, this property follows from Lebrun's (2006) equilibrium characterization. For completeness, we provide a new and self-contained proof.

Lemma 1 *If $\bar{v}_i = \bar{v}_j$ then $\bar{b}_i = \bar{b}_j$. If $\bar{v}_i > \bar{v}_j$ then $\bar{b}_i \geq \bar{b}_j$.*

Proof. See the Appendix. ■

The focus of the paper is on the possibility that $\bar{b}_i > \bar{b}_j$. The current section describes some first theoretical insights. We stress that much of the material in this section (specifically Lemmata 1 and 2 and thus by extension Proposition 1) follows from Lebrun's (2006) work. Rather than merely invoking his results, we have opted to develop the key insights at a slower pace to make more explicit what the driving forces of bid-separation are. Thus, our proofs of the results are new and self-contained. Moreover, our way of thinking about bid-separation is, at least cosmetically, conceptually different from Lebrun's (2006). We try to directly identify the set of types of "strong" bidders that separate from their competitors. Lebrun

⁵That is, the reverse hazard rate $\frac{f_i(v)}{F_i(v)}$ is strictly decreasing on an interval $(\underline{v}_i, \underline{v}_i + \delta)$, $\delta > 0$. The assumption can be further weakened if \underline{v}_i is not the same for all i .

(2006) instead reformulates the problem by imagining that bidders' type supports are extended in such a way that bidders have what looks like a common maximum bid. However, types on the extended part of the support are realized with probability zero and so bidders may in reality have distinct maximum bids (that is, the intervals on which different bidders bid with probability one may not coincide). In other words, bid-separation is taking place "behind the scenes" in Lebrun (2006). We choose a formulation that makes bid-separation take centre stage.

Let $\varphi_i(b)$ denote bidder i 's inverse bidding strategy. Assume, for now, that there is a range of bids where all bidders are active. If bidder i with type v contemplates bidding in this range, his expected payoff is $(v - b) \prod_{j \neq i} F_j(\varphi_j(b))$, which is maximized where

$$\ln(v - b) + \sum_{j \neq i} \ln F_j(\varphi_j(b))$$

is maximized. Deriving the first order condition and imposing the equilibrium condition that $v = \varphi_i(b)$ produces

$$\sum_{j \neq i} \frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b}. \quad (1)$$

Summing (1) across all agents and subtracting (1) for agent i yields the system of differential equations

$$\frac{d}{db} \ln F_i(\varphi_i(b)) = \frac{1}{n-1} \left[\sum_{j \neq i} \frac{1}{\varphi_j(b) - b} - \frac{n-2}{\varphi_i(b) - b} \right]. \quad (2)$$

It follows from Lemma 1 that if $\bar{v}_i = \bar{v}$ for all i , then there is some \bar{b} such that $\varphi_i(\bar{b}) = \bar{v}_i$, $i = 1, \dots, n$. Hubbard and Paarsch (2014) discuss this system at length and compares ways in which researchers have gone about numerically solving this system which rarely admits a closed-form solution. As we have noted, the approaches considered are generally valid only if $n = 2$ or if $\bar{v}_i = \bar{v}$ for all bidders $i = 1, 2, \dots, n$.

However, the point of the current paper is to allow $n > 2$ and $\bar{v}_i \neq \bar{v}_j$ for some (i, j) pair. For concreteness, and in line with the relevant empirical literature, assume in the remainder that bidders draw types from one of two distribution functions. Bidders $1, \dots, m_1$ draw types from the distribution F_1 , while bidders $m_1 + 1, \dots, n$ all draw types from F_n . Let $m_n = n - m_1$ denote the number of bidders in the latter group. It

follows from Lebrun (2006) that bidders within each of the two groups use symmetric strategies. Thus, the first m_1 bidders all use the same inverse bidding strategy as bidder 1, $\varphi_1(b)$. Likewise, the last m_n bidders are all ex ante identical to bidder n , and thus they all use the strategy $\varphi_n(b)$.

The complete solution to an asymmetric auction model requires appropriate boundary conditions. As such, we continue with an overview of these conditions.

First, by Lemma 1, $\bar{b}_1 = \bar{b}_n$ if $\bar{v}_1 = \bar{v}_n$. Of course, the case where $\bar{v}_1 \neq \bar{v}_n$ is the most interesting for our purposes. Thus, assume $\bar{v}_1 > \bar{v}_n$. Then, $\bar{b}_1 \geq \bar{b}_n$. Nonetheless, all bidders must share the same maximum bid if $m_1 = 1$. The reason is familiar; if $\bar{b}_1 > \bar{b}_n$, then bidder 1 with type \bar{v}_1 could lower his bid slightly and still win with probability one. Hence, assume from now on that $m_1 \geq 2$. The boundary conditions are that $\varphi_i(\bar{b}_i) = \bar{v}_i$, $i = 1, n$. Moving forward, however, we further assume that the supports strictly overlap, such that $[\underline{v}_1, \bar{v}_1] \cap [\underline{v}_n, \bar{v}_n]$ has strictly positive measure. With no overlap, or $\underline{v}_1 \geq \bar{v}_n$, the existence of the last m_n bidders is irrelevant as they would have no chance of winning the auction at a price that does not exceed their valuation. In other words, the problem is interesting only if the supports overlap. Hence, we have $\bar{v}_1 > \bar{v}_n > \underline{v}_1, \underline{v}_n$. For future reference, define $\underline{v} = \max\{\underline{v}_1, \underline{v}_n\}$.

Second, bidding behavior among low types is important as initial conditions are derived from this. Let \underline{b} denote what Lebrun refers to as the lowest “serious bid”. Only bids above \underline{b} has a positive probability of winning in equilibrium. The easiest cases are when (i) $m_1, m_n \geq 2$ or when (ii) $m_1 \geq 2$, $m_n = 1$, and $\underline{v}_1 \geq \underline{v}_n$. Then, \underline{b} simply coincides with \underline{v} and the initial condition is that $\varphi_i(\underline{v}) = \underline{v}$, $i = 1, n$. In either case, there are at least two bidders with valuations above \underline{v} and competition between these force the price up to at least \underline{v} . Bidders with valuations below \underline{v} are priced out of the market and do not compete. The remaining case in which (iii) $m_1 \geq 2$, $m_n = 1$, and $\underline{v}_n > \underline{v}_1$ is trickier, but Maskin and Riley (2000) and Lebrun (2006) have characterized the equilibrium value of \underline{b} in this case too. Here, $\underline{b} \in (\underline{v}_1, \underline{v}_n)$ and the initial condition is that $\varphi_n(\underline{v}_n) = \underline{b}$ and $\varphi_1(\underline{b}) = \underline{b}$. For convenience and completeness, Appendix B reviews and explains the initial conditions for all three cases in detail.

Figure 1 illustrates inverse bidding strategies when bid-separation occurs, i.e. when $\bar{b}_1 > \bar{b}_n$. The downward-sloping line will be defined momentarily and will be shown to play a crucial role. As denoted in Figure 1, let $\hat{v} = \varphi_1(\bar{b}_n) \in (\underline{v}_1, \bar{v}_1]$ be the endogenously determined type of bidder 1 that submits a bid of exactly \bar{b}_n . Going forward, it is important to realize that $\bar{b}_1 > \bar{b}_n$ if and only if $\hat{v} < \bar{v}_1$.

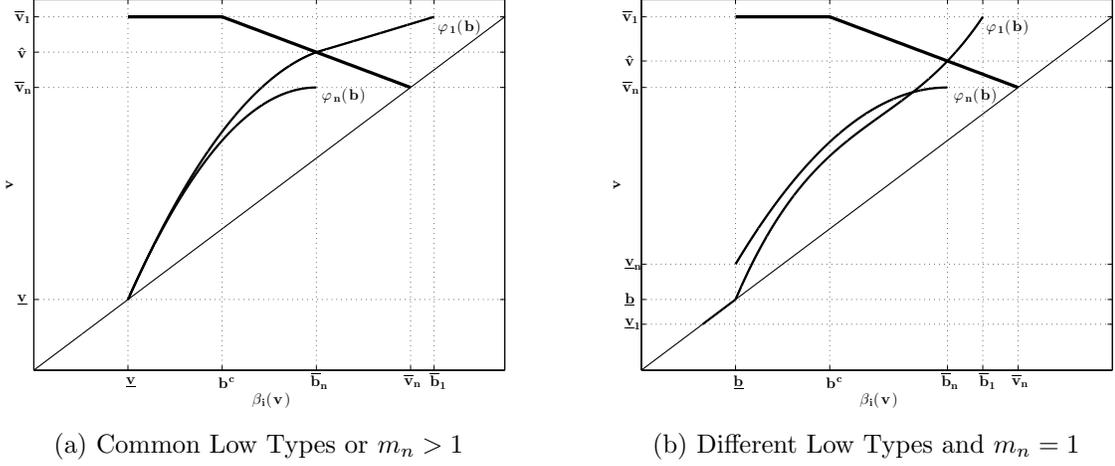


Figure 1: Potential Inverse Bid Functions

With the boundary and initial conditions in place, we return to “interior” bids. Since strategies are continuous and strictly increasing, it follows that for each bid in $[\underline{b}, \bar{b}_i]$, bidder i has a type that submit such a bid. Thus, $[\underline{b}, \bar{b}_1] = [\underline{b}, \bar{b}_n] \cup [\bar{b}_n, \bar{b}_1]$ consists of one interval of bids where all bidders are potentially active as well as a (possibly empty) set of higher bids where only bidders in the first group compete. The system in (2) still accurately describes behavior for bids in $(\underline{b}, \bar{b}_n)$. If $\bar{b}_n < \bar{b}_1$, then (2) should be replaced by

$$\frac{d}{db} \ln F_1(\varphi_1(b)) = \frac{1}{m_1 - 1} \frac{1}{\varphi_1(b) - b} \quad (3)$$

for $b \in (\bar{b}_n, \bar{b}_1]$, since only the first m_1 bidders are active on this range. The simple form of (3) reflects the fact that at high bids the auction is essentially a symmetric auction (involving only a symmetric subset of the population).

As mentioned previously, in cases where $\bar{b}_1 = \bar{b}_n = \bar{b}$ (e.g., if either $\bar{v}_1 = \bar{v}_n$ or $m_1 = m_n = 1$) researchers typically solve for a pair $(\varphi_1(b), \varphi_n(b))$ of inverse bidding strategies along with the common high bid \bar{b} (which is unknown a priori) satisfying the system (2) as well as the initial conditions. Generalizing this to the case in which bid supports may potentially differ initially seems harder since the system (2) remains, but now we need to solve for not just a single unknown value (\bar{b}), but three $(\hat{v}, \bar{b}_n, \bar{b}_1)$. There are at least two ways of attacking this problem. Lebrun (2006) extends the

type support of the last m_n bidders in such a way that all bidders share the same maximum bid (\bar{b}) and the system in (2) applies globally. However, $f_n(v) = 0$ at $v > \bar{v}_n$ such that there is potentially a “kink” in (2) at \bar{b}_n . Nevertheless, conceptually the approach is the same as in the standard literature; make a guess on \bar{b} and use that in the extension of (2) to obtain a candidate solution. Hence, everything is once again determined by the single variable \bar{b} .

Our approach is different, although it of course yields the same answer. We reduce, in two steps, the three unknowns $(\hat{v}, \bar{b}_n, \bar{b}_1)$ to a single unknown. First, it is easy to reduce the three variables to two free variables. For instance, given any (\bar{b}_n, \bar{b}_1) pair, \hat{v} can be obtained by integrating (3) backwards (which can be done analytically) from \bar{b}_1 to infer the v_1 value (\hat{v}) for which $v_1 = \varphi_1(\bar{b}_n)$. Alternatively, for any (\hat{v}, \bar{b}_n) pair, \bar{b}_1 can be computed simply by integrating (3) forwards from $b = \bar{b}_n$, $\varphi_1(\bar{b}_n) = \hat{v}$. Although it may seem natural to consider finding a (\bar{b}_n, \bar{b}_1) pair (this would certainly be closer in spirit to Lebrun’s (2006) approach), we find it more fruitful to think of (\hat{v}, \bar{b}_n) as the pair to be determined.

Next, it turns out that there is a one-to-one mapping between \bar{b}_n and \hat{v} which means that the unknown pair can in fact be reduced to a one-dimensional problem. In other words, there is in reality only one free variable in the triplet $(\hat{v}, \bar{b}_n, \bar{b}_1)$. It is worth emphasizing the point that, because of this relationship, conceptually and computationally the problem is now no more complicated than in the models where $\bar{b}_1 = \bar{b}_n$ is known to hold at the outset. In both cases, the system (2) applies and one value is unknown beforehand—in the standard case this is \bar{b} , while in the general case it is \bar{b}_n (which implies a value for \hat{v} and in turn a value for \bar{b}_1). The chief difference is that in the standard models \hat{v} is known to equal \bar{v}_1 , whereas here we have to determine \hat{v} through a step that luckily turns out to be trivial.

To see this relationship, first note that Lebrun (2006) has shown that if $b \in (\underline{v}, \bar{b}_i)$ then $\varphi'_i(b) > 0$. The key step is now to show that if $\bar{b}_n < \bar{b}_1$ then the (left-)derivative of φ_n at \bar{b}_n is zero, $\varphi'_n(\bar{b}_n) = 0$. Note that this property makes a bidder’s problem “smooth” around \bar{b}_n , the pivotal point where a smaller bid would mean he competes against all bidders and a higher bid means he competes only against bidders in the first group. Recall that if $\bar{b}_n < \bar{b}_1$ then $\hat{v} < \bar{v}_1$.

Lemma 2 *If $\bar{b}_n < \bar{b}_1$ then $\varphi'_n(\bar{b}_n) = 0$.*

Proof. See the Appendix. ■

Since $\varphi_n(\bar{b}_n) = \bar{v}_n$ and $\varphi_1(\bar{b}_n) = \hat{v}$, it follows from (2) that

$$\text{sign}\{\varphi'_n(\bar{b}_n)\} = \text{sign}\left\{\frac{m_1}{\hat{v} - \bar{b}_n} - \frac{m_1 - 1}{\bar{v}_n - \bar{b}_n}\right\}. \quad (4)$$

Starting with the assumption that $\bar{b}_n < \bar{b}_1$, Lemma 2 makes it possible to solve for \hat{v} ,

$$\hat{v} = \frac{m_1}{m_1 - 1}\bar{v}_n - \frac{1}{m_1 - 1}\bar{b}_n. \quad (5)$$

Note that since any equilibrium candidate must satisfy $\bar{b}_n < \bar{v}_n$, it holds that $\hat{v} > \bar{v}_n$. Of course, the restriction that \hat{v} cannot exceed \bar{v}_1 must also be taken into account. The solution in (5) satisfies $\hat{v} < \bar{v}_1$ if and only if

$$\bar{b}_n > m_1\bar{v}_n - (m_1 - 1)\bar{v}_1, \quad (6)$$

the right hand side of which is smaller than \bar{v}_n whenever $\bar{v}_n < \bar{v}_1$. Indeed, from (2), if $\bar{b}_n \in (m_1\bar{v}_n - (m_1 - 1)\bar{v}_1, \bar{v}_n)$ then $\varphi_1(\bar{b}_n) = \bar{v}_1$ would imply $\varphi'_n(\bar{b}_n) < 0$, which is inconsistent with an equilibrium. Thus, for \bar{b}_n candidates in this range, it *must* be the case that $\bar{b}_n < \bar{b}_1$, meaning that \hat{v} is determined by (5). Values of \bar{b}_n for which $\bar{b}_n \geq \bar{v}_n$ are inconsistent with an equilibrium. This leaves the possibility that $\bar{b}_n \leq m_1\bar{v}_n - (m_1 - 1)\bar{v}_1$, at least in the case where the right hand side exceeds \underline{v} . However, if $\bar{b}_n < m_1\bar{v}_n - (m_1 - 1)\bar{v}_1$ then the candidate in (5) would exceed \bar{v}_1 . Stated differently, there are no feasible values of $\varphi_1(\bar{b}_n)$ for which $\varphi'_n(\bar{b}_n)$ is not strictly positive. Then, by Lemma 2, any such candidate must thus satisfy $\bar{b}_n = \bar{b}_1$, or equivalently $\hat{v} = \bar{v}_1$. In summary, \hat{v} has been characterized for each \bar{b}_n candidate, $\bar{b}_n \in (\underline{b}, \bar{v}_n)$. Returning to the situations presented in the panels of Figure 1, the piecewise linear function in each subplot depicts this relationship. Note that the intersection of this line with the inverse bidding strategy $\varphi_1(b)$ obtains at the point (\bar{b}_n, \hat{v}) in both figures. We state this insight formally as a proposition.

Proposition 1 *For any $\bar{b}_n \in (\underline{b}, \bar{v}_n)$, \hat{v} is uniquely determined by*

$$\hat{v} = \min\left\{\bar{v}_1, \frac{m_1}{m_1 - 1}\bar{v}_n - \frac{1}{m_1 - 1}\bar{b}_n\right\}. \quad (7)$$

Proof. In text. ■

Proposition 1 is what simplifies the dimensionality of this general problem, making

the complexity of its solution akin to that of the common bid support setting as we suggested earlier. In the general case, there are evidently two possibilities: either $\hat{v} = \bar{v}_1$ (no bid-separation) or $\hat{v} < \bar{v}_1$ (bid-separation occurs). Given (7), the latter holds if and only if \bar{b}_n exceeds the critical value b^c , where

$$b^c = m_1 \bar{v}_n - (m_1 - 1) \bar{v}_1, \quad (8)$$

which is indicated in the subplots of Figure 1 as the kink points in the piecewise functions. If $\bar{b}_n \leq b^c$, then $\hat{v} = \bar{v}_1$ and there is no bid-separation. As such, there are precisely two mutually exclusive possibilities: either (i) $\bar{b}_1 = \bar{b}_n \leq b^c$, or (ii) $\bar{b}_1 > \bar{b}_n > b^c$.

In equilibrium, $\bar{b}_n > \underline{b}$. Thus, if $\underline{b} > b^c$ then $\bar{b}_n > \underline{b} > b^c$ must hold. In this case, bid-separation must occur in equilibrium. Stated differently, $\underline{b} > b^c$ is sufficient for bid-separation (but not necessary).

Corollary 1 *In equilibrium, $\bar{b}_n < \bar{b}_1$ if $\underline{b} > b^c$.*

Proof. In text. ■

Corollary 1 reveals, as is intuitive, that bidding must break into two regions, or $\bar{b}_n < \bar{b}_1$, if \bar{v}_1 is sufficiently large compared to \bar{v}_n . Note also that b^c is decreasing in m_1 . In other words, bid-separation must occur if m_1 is large enough. The intuition behind the link between the number of bidders and bid-separation is explained in more detail in Section 3.

A complementary sufficient condition for bid-separation can be derived. To set the scene, consider the example from the introduction in which two billionaire art collectors compete against two art students. Let \bar{b}_1^s denote the maximum bid in a (counterfactual) symmetric auction involving only the billionaires. It is a standard exercise to obtain \bar{b}_1^s as it is derived from a symmetric auction. That is, \bar{b}_1^s need not be known to solve the (in this symmetric case) equation characterizing equilibrium behavior. Imagine for the moment that $\bar{b}_1^s > \bar{v}_n$, where \bar{v}_n denotes the highest possible type among the art students. Once the students enter the auction, it seems unlikely that the billionaires will accommodate them by lowering their maximum bid. In this case, bid-separation must occur, as even a student of type \bar{v}_n will be unwilling to bid above \bar{b}_1^s . This style of argument can be further refined. Thus, it turns out that bid-separation must take place if \bar{b}_1^s exceeds b^c . The reason is that $\bar{b}_1 > \bar{b}_1^s$; i.e., the

highest bid increases when the last m_n bidders enter the auction. Thus, $\bar{b}_1 > \bar{b}_1^s > b^c$. However, as noted after (8), $\bar{b}_1 > b^c$ occurs only in the case of bid-separation.

Corollary 2 *In equilibrium, $\bar{b}_n < \bar{b}_1$ if $\bar{b}_1^s > b^c$.*

Proof. See the Appendix. ■

Note that if $\underline{v}_1 > \underline{v}_n$ then $\bar{b}_1^s > \underline{v}_1 = \underline{b}$, in which case Corollary 2 is stronger than Corollary 1.

3 The incidence of bid-separation

The current section is devoted to comparative statics on the incidence of bid-separation. First, given distributions (F_1, F_n) , we show that bid-separation is more likely the more bidders there are in either group. We then consider the effects of changing the distributions, making bidders “stronger”. Two ways of modeling increased strength is examined, both of which imply that the bidder is more likely to realize a high type. In one, the type support stays unchanged, while in the other the support is stretched. In either case, bid-separation is more likely to occur when bidders in the first group grow stronger. However, the same may or may not hold when bidders in the second group become stronger. The comparative statics described above are qualitative in nature. The section concludes with a specific example in which it is analytically possible to make quantitative predictions. Section 4 details how simulations can assist in deriving comparative statics in more general parameterized problems.

There are substantial technical difficulties involved in analyzing asymmetric first-price auctions in general, some of which come into play in a few of the following comparative statics exercises. Lebrun (2006, footnote 8) describes the potential pitfalls stemming from the fact that $\ln F_i(\varphi_i(b))$ tends to $-\infty$ if b tends to \underline{v}_i . Lebrun (2006) cleverly solves these issues, at the cost of a somewhat more complicated proof technique. Here, when these issues arise we settle for the simpler proofs that can be constructed with the assumption that there is a binding reserve price (r) in place, with $r \in (\underline{v}, \min\{\bar{v}_1, \bar{v}_n\})$. This assumption greatly simplifies the proofs and seems conceptually to come at little cost, since r could be arbitrarily close to \underline{v} . The system of differential equations must now satisfy $\varphi_i(r) = r$, $i = 1, n$ whenever $m_1, m_n \geq 2$ (Lebrun (2006)). The technical significance of the assumption is that $\ln F_i(\varphi_i(b))$ is finite even as $b \rightarrow r$. Lebrun’s (2006, Section 3) simpler proof technique can then

be adapted to prove some of the result in the current subsection. Moreover, Lebrun (2006) also establishes that equilibrium is unique when the reserve price is binding.

3.1 Varying the number of bidders

The first comparative statics exercise holds the distributions (F_1, F_n) fixed but varies the number of bidders. Thus, consider two auctions with different numbers of participants. Let (m_1, m_n) and (m'_1, m'_n) denote the number of bidders in the first and second auction, respectively, with $m'_1 \geq m_1 \geq 2$ and $m'_n \geq m_n \geq 2$. Hence, there are more bidders of either kind in the second auction than in the first auction. Note that the relationship in Proposition 1 depends on m_1 but not on m_n . Let \hat{v} and \hat{v}' denote strong bidders' equilibrium cut-off type in the two auctions. Recall that if \hat{v} decreases then the interval of separating types, $[\hat{v}, \bar{v}_1]$, increases. Thus, bid-separation becomes more pronounced.

Proposition 2 *Assume $m'_1 \geq m_1 \geq 2$, $m'_n \geq m_n \geq 2$ and $m'_1 + m'_n > m_1 + m_n$. Assume there is a binding reserve price in place, with $r \in (\underline{v}, \min\{\bar{v}_1, \bar{v}_n\})$. Then, $\hat{v}' \leq \hat{v}$ with $\hat{v}' < \hat{v}$ if $\hat{v} < \bar{v}_1$. Consequently, if bid-separation occurs under (m_1, m_n) then it also occurs under (m'_1, m'_n) , i.e. as the number of bidders increases.*

Proof. See the Appendix. ■

Proposition 2 is fairly intuitive. First, \hat{v} weakly decreases as m_n increases. To understand the intuition it is easiest to consider the limit as $m_n \rightarrow \infty$. Competition is so intense among the weak group that these bidders will bid close to their true type. The strong bidders are concerned with the highest type among the weak bidders. As $m_n \rightarrow \infty$, this type approaches \bar{v}_n . Thus, from the strong bidders' point of view, \bar{v}_n eventually takes a role similar to a reserve price; a bid below \bar{v}_n results in a loss with near-certainty. Consequently, \hat{v} must decrease, and eventually approach \bar{v}_n . Similarly, \hat{v} is weakly decreasing in m_1 because the more intense competition within the strong group spurs more aggressive bidding behavior, such that more strong types separate away from the weak group.

3.2 Changing type distributions

In the following we keep the number of bidders fixed but allow (F_1, F_n) to change. In keeping with the literature, we ask what happens as a group of bidders grow

“stronger” in some sense. An issue that arises immediately is whether the support of the distributions might change as well. For simplicity, we keep \underline{v}_i fixed but let \bar{v}_i vary. In principle, one can now decompose a change of a distribution into two steps; one step “stretches” the distribution to make it fit onto its new support and the second step then “bends” the stretched distribution to arrive at the final distribution. Thus, we consider such “stretching” and “bending” separately, before finally bringing the two together. Of course, it remains to properly define those two notions.

Recall from (2) that $\frac{d}{db} \ln F_i(\varphi_i(b))$ or $\frac{f_i(\varphi_i(b))}{F_i(\varphi_i(b))} \varphi_i'(b)$ plays a crucial role in determining equilibrium. The ratio $\frac{f_i(v)}{F_i(v)}$ appears routinely in comparative statics in auction theory. It is known as the reverse hazard rate. When we “stretch” a distribution we will do so in a manner that keeps the reverse hazard rate unchanged on the old support. When we “bend” a distribution we do so by changing the reverse hazard rate. We begin by considering the latter exercise.

3.2.1 Bending the distributions

Fix the size of the two groups, $m_1, m_n \geq 2$ and the supports $[\underline{v}_1, \bar{v}_1]$ and $[\underline{v}_n, \bar{v}_n]$, respectively. We let the pair of distributions change from (F_1, F_n) to (G_1, G_n) . Recall that G_i (strictly) dominates F_i in terms of the reverse hazard rate if

$$\frac{g_i(v)}{G_i(v)} > \frac{f_i(v)}{F_i(v)} \text{ for all } v \in (\underline{v}_i, \bar{v}_i].$$

Borrowing from Lebrun (1998), we will write $G_i \succ F_i$ if the above holds. Here, we will assume that G_i either strictly dominates F_i in terms of the reverse hazard rate or is identical to F_i . Borrowing from Lebrun (1998) once again, this will be denoted $G_i \succeq F_i$.

Let (\hat{v}^F, \bar{b}_n^F) and (\hat{v}^G, \bar{b}_n^G) denote the equilibrium values of (\hat{v}, \bar{b}_n) when the distributions are (F_1, F_n) and (G_1, G_n) , respectively. The characterization in Proposition 1 implies that \hat{v} is non-increasing with \bar{b}_n .

Proposition 3 *Assume $G_i \succeq F_i$, $i = 1, n$ and $m_1, m_n \geq 2$. Assume there is a binding reserve price in place, with $r \in (\underline{v}, \min\{\bar{v}_1, \bar{v}_n\})$. Then, $\bar{b}_n^G \geq \bar{b}_n^F$ and thus $\hat{v}^G \leq \hat{v}^F$. Consequently, if bid-separation occurs under (F_1, F_n) it also occurs under (G_1, G_n) ; i.e., when bidders become stronger.*

Proof. See the Appendix. ■

Proposition 3 is consistent with intuitive comparative statics in Lebrun (1998). With two groups of bidders who share the same support, Lebrun (1998) shows that the common maximum bid must increase when one group of bidders become stronger in the sense of reverse hazard rate dominance. Applied to our setting, this would suggest that if we hold \hat{v} fixed at \hat{v}^F , then \bar{b}_n should increase when (F_1, F_n) is replaced by (G_1, G_n) . Given the inverse relationship between \bar{b}_n and \hat{v} (see Proposition 1 or the piecewise linear functions in Figure 1), it is then no surprise that the latter decreases.

Recall that the only structure we have imposed on the relationship between F_1 and F_n is that $\bar{v}_1 > \bar{v}_n$. Thus, despite the language of strong and weak groups, F_1 need not dominate F_n in the sense of reverse hazard rate dominance or even in the sense of first order stochastic dominance. It is, for instance, entirely possible that the mean of F_n is the same or higher than that of F_1 . The latter distribution could simply be more dispersed or spread out than the former.

However, it is standard in the theoretical literature to assume that group 1 is stronger than group n in a stricter sense. In particular, it is often assumed that F_1 dominates F_n in terms of the reverse hazard rate. This implies that $F_n(v) > F_1(v)$ for all $v \in (\underline{v}_n, \bar{v}_n]$. If G_n strictly dominates F_n , it holds that $F_n(v) > G_n(v)$ for all $v \in (\underline{v}_n, \bar{v}_n)$ as well. Thus, if G_n strictly dominates but is “close to” F_n , the asymmetry between bidders diminishes in the sense that $F_n(v) - F_1(v) > G_n(v) - F_1(v) > 0$ for all $v \in (\underline{v}_n, \bar{v}_n)$. One implication of Proposition 3 is now that reducing the asymmetry does not necessarily lead to less bid-separation. In other words, a larger mass of group 1 types may separate away from the second group of bidders.

3.2.2 Stretching the distributions

Next, we allow \bar{v}_i to change. Let $\bar{v}'_i \geq \bar{v}_i$ denote the new upper bound of the support. Let bidder i 's original distribution be denoted F_i and his new distribution be denoted H_i . In Maskin and Riley's (2000) terminology, the latter is a stretched version of the former in the sense that for some $\lambda \in (0, 1)$, the new distribution takes the form $H_i(v) = \lambda F_i(v)$ on $v \in [\underline{v}_i, \bar{v}_i]$ (and is continuously extended on $(\bar{v}_i, \bar{v}'_i]$). Equivalently, F_i is a truncation of H_i on a smaller interval. Evidently,

$$\frac{h_i(v)}{H_i(v)} = \frac{f_i(v)}{F_i(v)} \text{ for all } v \in [\underline{v}_i, \bar{v}_i].$$

That is, the reverse hazard rate is unchanged on the original support. Assume first that F_1 is stretched. Let \hat{v}^F and \hat{v}^H denote the equilibrium values of the threshold type before and after the change. For the proof of the following result, a binding reserve price is not needed (although we still assume a unique equilibrium).

Proposition 4 *Assume $m_1 \geq 2$, $m_n \geq 1$. If (F_1, F_n) is replaced by (H_1, F_n) then $\hat{v}^H = \hat{v}^F$ if $\hat{v}^F < \bar{v}_1$. That is, bid-separation becomes more pronounced if F_1 is stretched.*

Proof. The proposition follows from the observation that the old (and unique) solution on the common range of bids, $[\underline{v}, \bar{b}_n]$, remains the solution when F_1 is replaced by H_1 . The reason is that the system in (2) does not change when F_1 is stretched, due to the fact that the reverse hazard rate is unchanged on $[\underline{v}_1, \bar{v}_1]$. It also follows immediately that if there is originally no bid-separation ($\hat{v}^F = \bar{v}_1$) then bid-separation may arise ($\hat{v}^H < \bar{v}'_1$) when F_1 is stretched. ■

Consider next the case where it is F_n that is stretched. To understand how \hat{v} changes with \bar{v}_n (or when F_n is replaced by H_n), it is useful to consider the extreme cases. If \bar{v}_n is close to zero, the weak group presents no threat to the strong group, and \hat{v} will itself be small. On the other hand, if \bar{v}_n is close to \bar{v}_1 , the asymmetry between the two groups almost disappears at high valuations. In this case, \hat{v} must be close to \bar{v}_n . Thus, there can be no bid-separation if F_n is stretched sufficiently far (unless $\bar{v}'_n > \bar{v}_1$ such that the roles of the two groups are reversed). In the next subsection we present an example where it can be directly verified that \hat{v} is monotonic in \bar{v}_n .

Combining Propositions 3 and 4 yields the unambiguous conclusion that if the strong group is made stronger, then bid-separation is more likely to arise. However, Proposition 3 and the observation in the previous paragraph imply that there are conflicting forces at play when the weak group is made stronger. Depending on which force dominates, bid-separation may become more or less pronounced.

3.3 The uniform model

Assume both distributions are uniform, on different supports. The uniform distribution satisfy Lebrun's (2006) condition, implying the equilibrium is unique. It turns out that the exact values of \hat{v} , \bar{b}_n , and \bar{b}_1 can be derived analytically. Thus, it is possible to describe precisely when $\bar{b}_n < \bar{b}_1$. The proof, in the appendix, may be of some

independent interest. The proof demonstrates that in the uniform case, insights from mechanism design theory can be used to obtain another, quite separate, characterization of the (\widehat{v}, \bar{b}_n) pair. Combined with the characterization above, it is then possible to solve explicitly for both \widehat{v} and \bar{b}_n . As discussed earlier, for any parameterization, it is then easy to derive \bar{b}_1 . To simplify the exposition, it is assumed that $\underline{v}_1 = \underline{v}_n = 0$. However, the proof is easily modified to handle $\underline{v}_1 \neq \underline{v}_n$. Likewise, we assume there is no reserve price, but it is possible to extend the characterization to allow for one.

For ease of notation in formulating the result, let $m = m_1 + m_n - 1 = n - 1$ denote the number of rivals faced by any bidder. Finally, define $\kappa(m_1, m_n)$ and $\tau(m_1, m_n)$, respectively, as

$$\kappa(m_1, m_n) = \frac{(m_1 + 1)m - \sqrt{(m_1 + 1)^2 m^2 - 4m_n m_1 m}}{2m_n},$$

$$\tau(m_1, m_n) = \frac{m_1 - \kappa(m_1, m_n)}{m_1 - 1},$$

and note that $\kappa(m_1, m_n) \in (0, 1)$ while $\tau(m_1, m_n) > 1$. Of course, both functions are independent of \bar{v}_1 and \bar{v}_n . It can be shown that $\kappa(m_1, m_n)$ is strictly increasing in both its arguments and that $\tau(m_1, m_n)$ is strictly decreasing in both its arguments.

The functions $\kappa(m_1, m_n)$ and $\tau(m_1, m_n)$ allow a complete characterization of the equilibrium (\widehat{v}, \bar{b}_n) pair. Hence, it is possible to identify precisely when bid-separation occurs. To put this result in context, Vickrey (1961) first considered a specific two-bidder example involving asymmetric uniform distributions.⁶ A half-century later, Kaplan and Zamir (2012) obtained a full, closed-form, equilibrium characterization that holds in any *two-bidder* first-price auction in which both distributions are uniform, allowing for arbitrary supports. The time gap between the two papers illustrates the magnitude of the difficulties involved in analyzing asymmetric auctions. To our knowledge, the next result is the first to address asymmetric auctions with more than two bidders in the uniform-distribution setting. A closed-form equilibrium characterization is not currently within reach, and may or may not be theoretically possible. However, it is not required in order to understand when bid-separation occurs.

Proposition 5 *Assume $F_i(v) = \frac{v}{\bar{v}_i}$, $v \in [0, \bar{v}_i]$, $i = 1, n$, with $\bar{v}_1 > \bar{v}_n > 0$. Assume $m_1 \geq 2$, $m_n \geq 1$. Equilibrium properties depend on the relative difference between*

⁶In fact, Vickrey (1961) assumed that one bidder's type is known, or that his type-distribution is degenerate.

supports, $\frac{\bar{v}_1}{\bar{v}_n}$:

1. If $\frac{\bar{v}_1}{\bar{v}_n} \leq \tau(m_1, m_n)$ (the supports do not differ too much), then both kinds of bidders share the same maximum bid, $\bar{b}_n = \bar{b}_1$ and $\hat{v} = \bar{v}_1$, with

$$\bar{b}_n = \frac{m}{\bar{v}_1 m_n + \bar{v}_n m_1} \bar{v}_1 \bar{v}_n.$$

2. If $\frac{\bar{v}_1}{\bar{v}_n} > \tau(m_1, m_n)$ (the supports differ considerably), then bid-separation occurs, $\bar{b}_n < \bar{b}_1$ and $\hat{v} < \bar{v}_1$, with

$$\bar{b}_n = \kappa(m_1, m_n) \bar{v}_n$$

and

$$\hat{v} = \tau(m_1, m_n) \bar{v}_n.$$

Moreover, the equilibrium is continuous in all the parameters, m_1 , m_n , \bar{v}_1 and \bar{v}_n .

Proof. See the Appendix. ■

Proposition 5 refines Corollary 1 in a special case.

Note that F_i is stretched in the sense of Proposition 4 whenever \bar{v}_i increases. Thus, the uniform model can be used to complement Proposition 2 and 4 with some quantitative insights.

In Figure 2, we consider bidders who draw valuations from uniform distributions for which we consider various combinations of (\bar{v}_i, \bar{v}_j) . The diagonal line from the southwest corner to the northeast corner is the symmetric bidder case ($\bar{v}_i = \bar{v}_j$). Northwest (southeast) of this line, bidders of class j (i) are the strong bidders as the highest possible valuation for these bidders exceeds that of bidders from the rival class. The various lines that are plotted correspond to the sufficient condition for bid-separation from Corollary 1 (which applies to any distribution) and the necessary condition from Proposition 5 (which is uniform-distribution-specific). In the white areas, the sufficient condition for bid-separation from Corollary 1 and the necessary condition from Proposition 5 are both satisfied and bid-separation occurs. In the darker shaded areas, the condition specified in Corollary 1 is not met, but bid-separation still occurs because the condition from Proposition 5 holds. Only for the lightly-shaded gray, cone-shaped area (partitioned by the line $\bar{v}_i = \bar{v}_j$) do all bidders tender the same high bid and bid-separation will not occur. Note that the regions themselves are asymmetric—this is because rather than reflect the same example over

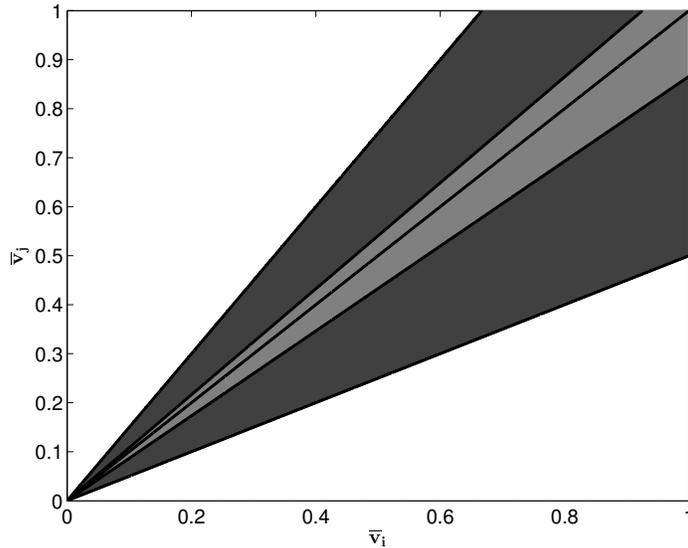


Figure 2: Bid-separation with $\underline{v}_1 = \underline{v}_n = 0$

the $\bar{v}_i = \bar{v}_j$ line, we always consider the case where $(m_i, m_j) = (2, 3)$ so that when bidders of class j are strong, there are three strong and two weak bidders, while the opposite is true when bidders of class i are strong. The figure helps visualize, within the context of examples concerning the uniform distribution, the magnitudes of the light gray region in which bid-separation does not occur (as researchers have most commonly assumed) as well as the region in which the sufficient condition for bid-separation from Corollary 1 is not met, yet bid-separation still obtains.

Recall that Proposition 2 implies that bid-separation is more likely as the number of bidders increases. The discussion around Proposition 4, on the other hand, suggests that bid-separation is more likely as \bar{v}_1 increases or \bar{v}_n decreases. In other words, bid-separation is more likely the higher \bar{v}_1/\bar{v}_n is. In Figure 3, we convey this for the case of asymmetric uniform distributions. The three-dimensional bar graph plots m_1 and m_n along the horizontal axes with $(\bar{v}_1/\bar{v}_n - 1)$ on the vertical axis. Specifically, the height of the bars indicate the maximum percentage that \bar{v}_1 can exceed \bar{v}_n without bid-separation occurring. As an example, the largest bar represents the most likely scenario for a common bid support—when $(m_1, m_n) = (2, 1)$ in which case if \bar{v}_1 exceeds \bar{v}_n by more than 23.61%, bid-separation must occur. Notice that regardless of whether there is an increase in the number of weak or strong bidders (or both), the height of the bars decrease rapidly. For instance, if (m_1, m_n) were instead $(3, 2)$,

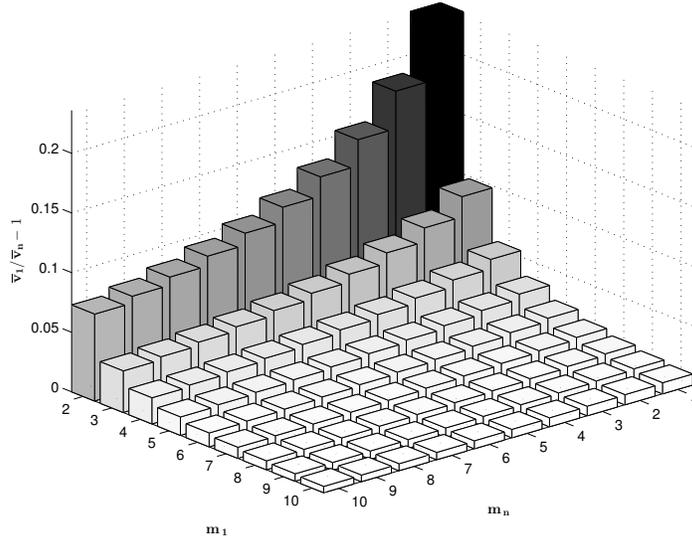


Figure 3: Prevalence of Bid-separation given Bidder Composition

only a 8.11% difference between \bar{v}_1 and \bar{v}_n would imply bid-separation.⁷ Holding the number of strong bidders fixed, the threshold difference falls slower as m_n increases than in the opposite case where the number of weak bidders is fixed and m_1 increases. There are 90 bars depicted and 74 of them take on a value less than 0.05, meaning if the strong bidders' high type exceeds the weak bidders' strong type by five percent, bid-separation will occur for all of those instances. If the magnitude considered is 0.02, 52 of the instances still result in bid-separation.

The figure helps convey the practical importance bid-separation. Recall the argument in the introduction that preferential treatment of one group of bidders is mathematically equivalent to rescaling the support. The uniform example thus demonstrates that even if the two groups are initially identical, even a modest preference rate could lead to bid-separation—meaning simulations, counterfactual experiments, and even structural estimation routines cannot be based on an assumption of a common maximal bid. Lastly, notice one implication of Corollary 1 is that in a general model, bid-separation must occur if $\frac{\bar{v}_1}{\bar{v}_n} > \frac{m_1}{m_1-1}$. If $m_1 = 2$, for instance, it is thus

⁷This partition of the five bidders is not unreasonable. For example, Krasnokutskaya and Seim (2011) report that on average there were 2.6 large bidders and 1.7 small bidders vying for contracts in their California procurement data. Moreover, on average 6.6 large firms and 3.9 small firms requested plans for the contracts which is typically the proxy empirical researchers use for the number of potential bidders.

sufficient that \bar{v}_1 is twice as large as \bar{v}_n (assuming $\underline{b} = 0$). The larger m_1 is (or the larger \underline{b} is), the lower the relative difference between the high types of each group in order to guarantee bid-separation. That said, these relative differences might be smaller for bid-separation to occur as the uniform example suggested—recall that the dark shaded areas in Figure 2 represent instances in which the high types were quite close, but the sufficient condition was not satisfied.

The example can also be used to illustrate Proposition 3. Note that any convex distribution function on $[0, \bar{v}_i]$ strictly dominates the uniform distribution on $[0, \bar{v}_i]$ in terms of the reverse hazard. Proposition 3 thus implies that if the distributions are convex, then bid-separation is more likely to occur (i.e., occurs for more (\bar{v}_1, \bar{v}_n) pairs) than in the uniform benchmark. Letting r approach zero then allows Proposition 5 to be used as a lower bound on the incidence of bid-separation.

4 Accounting for bid-separation in simulations

As we are the first to delve into the general asymmetric model with more than two bidders, researchers who have solved asymmetric auctions have typically been careful to limit attention to settings in which the standard boundary condition is sure to hold. Li and Riley (2007, footnote 8) explicitly note that they restrict attention to the case of identical supports. Instead, they argue that if the supports differ one could replace the distributions with approximations which do have a common support. Likewise, Hubbard and Paarsch (2014) claimed that boundary conditions might need adjustment in settings where the type supports differ and there are more than two bidders at auction, and so they explicitly restricted their analysis to the case of common type supports in trying to avoid such complications. These researchers, however, are among the few to explicitly recognize that bid supports may differ. That said, no one has proposed a solution technique that solves for the equilibrium of an asymmetric auction when bid-separation is possible.

As we have argued above, casual observation of auctions suggest bid-separation is a likely possibility in some settings and empirical researchers encounter raw data that suggests bidding supports may differ. Moreover, allowing for bid-separation is required to evaluate real-world policies such as bid preferences, where any avoidance of the bid-separation issue (in the spirit of the proposal by Li and Riley) would require approximations to the type distributions which would need to vary with the

preference rate. Given the potential for bid-separation in real world auctions, and the disadvantages that come with using approximations which map type distributions over different supports to ones on a common support in order to avoid the issue, we seek to strengthen the approach to solving an asymmetric auction.

There are no existing solution techniques that can directly accommodate settings that entail bid-separation. In the first subsection, we thus outline a two-step procedure for solving a general asymmetric auction that allows for bid-separation. The remainder of the section then describes each step in detail and provides examples that illustrate the significance of each step and of bid-separation in general.

4.1 A two-step solution approach

Unlike the symmetric first-price auction model which admits a closed-form solution for the Bayes–Nash equilibrium bid functions, asymmetric auctions rarely allow for such tractability. As such, researchers are left resorting to numerical methods to solve the system of differential equations under the appropriate boundary conditions. However, existing methods ignore bid-separation. Hubbard and Paarsch (2014) summarize the various ways in which researchers have gone about solving for the equilibrium. They can be broadly categorized as: (i) shooting algorithms, (ii) fixed-point iterations, and (iii) a polynomial approach. It is not our objective to delve into the merits of each of these approaches. Rather, we will provide a two-step solution method that first identifies whether bid-separation occurs and, if so, goes on to modify any existing solution approach – such as those just listed – to handle bid-separation.

The two steps are outlined next, with further details postponed to the following subsections:

1. *Identifying whether bid-separation occurs:* Begin by computing $b^c = m_1 \bar{v}_n - (m_1 - 1) \bar{v}_1$. Check if any of the sufficient conditions for bid-separation from Corollaries 1 or 2 apply; if so, move to step two. However, even if neither sufficient condition is met, bid-separation may still occur. To infer whether this is the case, recall that b^c provides an upper-bound on \bar{b} when the bid support is common to bidders and a lower-bound on \bar{b}_n when bid-separation obtains. Because of this, b^c can be used in a diagnostic test which allows for detecting whether bid-separation happens or not. Specifically, consider imposing the boundary conditions $\varphi_i(b^c) = \bar{v}_i$, $i = 1, n$, and integrating the system

(2) backwards under the assumption that the equilibrium involves a common bid support. One of two things will happen in integrating backwards:

- (a) The system blows up (becomes unbounded, going off to negative infinity) in approaching \underline{b} . We argue (see below) that this is evidence that all bids are below b^c in equilibrium—in particular, $\bar{b}_n = \bar{b}_1 = \bar{b} \leq b^c$ (bid-separation does not occur);
- (b) The inverse bid functions return values in the range $[\underline{v}, \bar{v}_n]$. We argue (see below) that this is evidence that some bidders tender amounts that exceed b^c in equilibrium—in particular, $\bar{b}_n, \bar{b}_1 \geq b^c$ (bid-separation occurs).

2. *Simulating equilibrium strategies:* Given the outcome of the first step, if the equilibrium involves bid-separation, solve the system (2) on the overlapping region of the bid support and then integrate (3) over the region that only class 1 bidders are active. In this case, (7) from Proposition 1 can be used to link \bar{b}_n and \hat{v} and to help delineate when (2) and (3) apply. If the equilibrium involves a common bid support, solve the system (2) over the entire support.

4.2 Step one: The Terminal Indicator test

To explain the test in the first step, recall the discussion that led to Figure 1 in Section 2. If any bid exceed b^c in equilibrium, then $\hat{v} < \bar{v}_1$ (bid-separation occurs) and, by Proposition 1, \hat{v} is pinned down by the downward sloping line in Figure 1. Similarly, if bids are all below b^c , then $\hat{v} = \bar{v}_1$ (bid-separation does not occur) and bidders share a common bid support. The test allows us to decipher whether bid-separation occurs or not because the solution to (2) is monotonic in \bar{b} . This is a critical feature, especially to researchers who adopt shooting algorithms as such behavior guides guesses for initial conditions across iterations; see, for example, Marshall, Meurer, Richard, and Stromquist (1994). Fibich and Gavish (2011) showed that the shooting algorithm is inherently unstable and the comparisons in Hubbard and Paarsch (2014) echo that this approach has issues.⁸ As such, we are not advocating an attempt to solve for the equilibrium by integrating backwards which would involve shooting as a basis for

⁸To be clear, the instability documented relates to shooting backwards from an ε -neighborhood of the true \bar{b} . In our case, we are interested in investigating behavior of the system when integrating backwards from the point b^c .

search of \bar{b}_n . We integrate backwards (once) not to actually approximate equilibrium, but merely to see whether the system blows up when starting from b^c or not.⁹ That is, it is precisely the tendency of the solution to blow up when fed a high bid that is too high that guides our diagnostic test.

Thus, to be explicit, if using (2) to shoot backwards from $\varphi_i(b^c) = \bar{v}_i$, $i = 1, n$, makes the solution blow up, then the conclusion is that $\bar{b} = b^c$ is just too high a guess. Hence, the true maximal bid is lower. In other words, $\bar{b}_n = \bar{b}_1 = \bar{b} \leq b^c$ and bid-separation does not occur. Conversely, if the solution does not blow up, then $\bar{b} = b^c$ is too small a guess. This in turn implies that $\bar{b}_n, \bar{b}_1 \geq b^c$ and so bid-separation occurs. Moving forward, we define an indicator variable, referred to as the “Terminal Indicator”, that equals one if the diagnostic test suggests bid-separation, and zero if it suggests that the bid support is common for all bidders.

To consider the accuracy of the diagnostic test, consider the plots depicted in Figure 4. In the top panel, we consider asymmetric uniform auctions in which $(m_1, m_n) = (3, 2)$ where we fix $\bar{v}_1 = 1$ but allow \bar{v}_n to vary. For each value of \bar{v}_n , we derive the Terminal Indicator. For comparison, we also report threshold values implied by Corollary 1 and Corollary 2. Specifically, for all points to the left of “Cor 1 Separation Indicator”, the condition of Corollary 1 is satisfied. For all points the left of “Cor 2 Separation Indicator”, the condition of Corollary 2 is satisfied. Note that $\underline{b} = 0$ here so Corollary 2 is stronger than Corollary 1. In addition, we denote whether the necessary and sufficient condition of Proposition 5 is satisfied. For every \bar{v}_n , the Terminal Indicator matches the prescription of Proposition 5.¹⁰

In the middle panel, we consider a situation in which bidders draw types from truncated normal distributions. Again, consider $(m_1, m_n) = (3, 2)$ and fix $\bar{v}_1 = 1$ while letting \bar{v}_n vary. In this example, we also fix the parameters of group 1’s distribution to be a normal distribution with mean 0.5 and standard deviation 0.25 which is truncated over the $[0, 1]$ support. For every \bar{v}_n , we consider the group receives valuations from a normal distribution with mean $\mu_n = \bar{v}_n/2$, has a fixed standard deviation, and is truncated over $[0, \bar{v}_n]$. In this figure, we indicate the threshold point

⁹As an aside, there may be a way to modify the fixed-point iteration approach of Fibich and Gavish (2011) so that it is capable of diagnosing whether bid-separation occurs or not. It would seem that one would need to embed flexibility in the boundary condition on the tying function, $t(v)$, that describes the type of bidder 1 that ties with (bids the same as) bidder n with type v to allow for the possibility that bidding supports may differ—so $t(\bar{v}_n)$ need not equal \bar{v}_1 .

¹⁰In constructing this figure, we consider a grid of \bar{v}_n values in which points are 0.0005 apart from each other.

at which the Terminal Indicator switched from predicting bid-separation to predicting a common support for three parameterizations, corresponding to different values of the variance for group n 's distribution. The condition of Corollary 1 is independent of the parameters of the distributions, depending only on the supports and number of players. As such, this threshold is fixed as group n 's distribution changes. Because we have fixed the distribution of group 1, Corollary 2 is also fixed across examples. Note that, although we did not require it in this exercise, the step 4 test always predicts bid-separation when these sufficient conditions are met. Moreover, the predictions change in reassuring ways—when the σ_n is low, the terminal indicator approaches the Corollary 2 threshold while, when the σ_n is high the terminal indicator approaches the uniform distribution threshold. When group n 's distribution has a lower variance, high types of group 1 need not worry as much about facing competitive group n bidders for a given \bar{v}_n . As such, competition within group n is what disciplines bidding behavior in a way that is similar to the symmetric auction setting involving only m_1 bidders. Likewise, when the variance of the group n distribution is high, the density becomes very flat, mimicking the uniform distribution. Though the comparison to the uniform case is not valid (remember that group 1's distribution is a fixed truncated normal distribution), we see that bid-separation approaches the uniform threshold. In fact, if the standard deviation of both groups is set sufficiently high, the predictions of the diagnostic test are exactly the same as in the uniform distribution case—the Terminal Indicator line falls directly on the Proposition 5 indicator depicted to the far right in the middle panel figure.

Lastly, consider the bottom panel. In this case, we again consider $(m_1, m_n) = (3, 2)$ bidders with types drawn from asymmetric truncated normal distributions over $[0, \bar{v}_i]$ where \bar{v}_n changes in the figure. Group 1's distribution is fixed to be a normal distribution with mean 0.5 and standard deviation 0.25 which is truncated over the $[0, 1]$ support. In this figure, $\sigma_n = 0.25$ for all parameterizations but we let μ_n vary. Specifically, we depict situations in which $\mu_n = \delta(\bar{v}_n - \underline{v}_n)$ where $\delta = \{1/4, 1/2, 3/4\}$. For a given \bar{v}_n , since σ_n is fixed, higher values for δ imply distributions that first-order stochastically dominate distributions associated with lower values of δ . Thus, by construction, we can investigate whether the diagnostic test respects the predictions of Proposition 3. Note that the figure in the right panel identifies the same relationship as this theoretical result—higher values of δ imply bid-separation occurs more often. Said another way, for a given \bar{v}_n , if bid-separation holds for a low value of δ , it is also

true that bid-separation occurs for a higher value of δ .

The above discussion gives us confidence that using the terminal indicator as a diagnostic test deciphers the correct type of equilibrium.

4.3 Step two: Simulating equilibrium strategies

We now turn to the second step in the procedure, approximating equilibrium strategies. We should note that our goal is not to take a stance on which solution technique is the best, but rather to make clear how the theory presented above can guide and modify the implementation of such algorithms.¹¹ If bidding is over a common support, the structure of the problem is exactly the same as the standard setting—though note, we can use the insight above to inform a better initial guess for \bar{b} which we know must be in the range $[\underline{b}, b^c]$.¹² On the other hand, if bid-separation occurs, the key input to our algorithm will be a guess on $\bar{b}_n \in [b^c, \bar{v}_n]$, paired with the theoretically consistent \hat{v} from Proposition 1, to approximate the system (2) over the intersection of the bid supports.

Here, we describe a solution technique which builds off the polynomial-based approach considered in Hubbard, Kirkegaard, and Paarsch (2013). Under this approach, the inverse bidding strategies are assumed to be polynomials, the coefficients of which are chosen to solve approximately the system of differential equations that characterize equilibrium behavior along with the unknown high bid such that conditions provided by theory (such as the appropriate boundary conditions) hold. Thus, assuming the equilibrium strategies can be approximated by Chebyshev polynomials,

¹¹For example, building on our comments in footnote 11, it may be possible to modify the fixed-point approach of Fibich and Gavish (2011) to handle bid-separation as well—the boundary conditions would need to be adjusted appropriately. For example, one could replace \bar{v}_1 in the boundary condition $t(\bar{v}_n) = \bar{v}_1$ with some $\hat{v} < \bar{v}_1$ guess, and then use the fixed-point algorithm to derive a candidate solution. If the solution does not return the value \bar{b}_n in Proposition 1, the \hat{v} guess is incorrect. One could then iterate on this procedure until a theoretically-consistent (\bar{b}_n, \hat{v}) pair is reached. Unfortunately, because the fixed-point approach involves a transformation it suppresses the one item, \bar{b}_n , about which we have some information (Proposition 1) which requires an iterative scheme to be written around the core fixed-point algorithm. In contrast, the approach we describe builds the \bar{b}_n - \hat{v} relationship directly into the algorithm.

¹²In the standard setting, the only information known a priori is that $\bar{b} \in [\underline{b}, \bar{v}_n]$.

the inverse bid function for bidder i can be expressed as

$$\hat{\varphi}_i(b; \boldsymbol{\alpha}_i, \bar{b}_n) = \sum_{k=0}^K \alpha_{i,k} T_k[x(b; \bar{b}_n)] \quad i = 1, n \quad (9)$$

where $x(\cdot)$ lies in the interval $[-1, 1]$ and where, for completeness, we have explicitly defined it as a transformation of the bid b under consideration.¹³ Here, $T_k(\cdot)$ denotes the k^{th} Chebyshev polynomial of the first kind and the vector $\boldsymbol{\alpha}_i$ collects the polynomial coefficients for bidder i . Thus,

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{k+1}(x) &= 2xT_k(x) - T_{k-1}(x) \quad k = 1, 2, \dots, K-1. \end{aligned}$$

We evaluate the system (2) over the Chebyshev nodes from the relevant bid support which have the property of minimizing the maximum interpolation error when approximating a function and so are often considered a good choice for the requisite grid. Casting the problem within the Mathematical Programming with Equilibrium Constraints (MPEC) approach proposed by Luo, Pang, and Daniel (1996) and advocated by Su and Judd (2012), the unknowns $(\boldsymbol{\alpha}_i, \bar{b}_n)$ are then chosen to minimize the deviations from the system (2) at the grid points such that the relevant theoretical (importantly, the boundary) conditions hold. Our focus has been on how the boundary conditions are modified if bidding supports differ. Once the boundary conditions are modified, our approach mirrors that of Hubbard, Kirkegaard, and Paarsch (2013) where readers can find an extended discussion of the shape-based constraints that guide search for the polynomial coefficients and high bid. Because Proposition 1 reduces the dimensionality of the problem so that instances involving bid-separation are no more complex than the common support setting, the approach generalizes naturally once the boundary conditions are modified with the exception that, under bid-separation, the inverse bid function of group n must satisfy the derivative condition of our Lemma 2 above.

The remainder of this section presents fully solved bidding strategies in a few examples. Continuing the tradition of Vickrey (1961) and Kaplan and Zamir (2012),

¹³We abuse notation slightly so that, in the case where bidders tender offers over a common support, $\bar{b}_n = \bar{b}$.

we first consider the uniform-distributions model where, because more than two bidders are considered, no closed-form solution exists. Then, continuing the example in Subsection 4.2, we move on to examples that involve truncated Normal distributions.

EXAMPLE 1: Consider an auction with $m_1 = 3$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 3/4]$. In this setting, $b^c = 1/4$. If the m_1 bidders were at a symmetric auction, the equilibrium strategy would prescribe that high-type bidders tender a bid of $\bar{b}^s = 2/3$. Thus, the condition of Corollary 2 holds and we know bid-separation will occur. Moreover, applying what we know from Proposition 5(b): $\bar{v}_1/\bar{v}_n = 4/3 > 1.0811 = \tau$ so the necessary and sufficient condition which applies to this example confirms that bid-separation must happen. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 5(a). Note that the group n inverse bid function satisfies the property of Lemma 2 as it flattens out so that its (left-)derivative is zero at \bar{b}_n . The line indicating the value for \hat{v} suggests that for about 19% of group 1 types, they compete only with other group 1 bidders. The inverse bid functions are nearly identical for low bids but as they approach \bar{b}_n there is a clear divergence.

For Example 1, we complement the correct solution with an illustration of how things look if the wrong boundary condition is imposed. Specifically, in the panel on the right we present the output of an approximation in which the common bid support conditions are imposed. There is a stark contrast across these figures and the wrong solution on the right should raise some immediate flags. Recall that a symmetric auction with m_1 bidders leads to high types tendering a bid of $\bar{b}^s = 2/3$; in fact the symmetric auction equilibrium involves a linear bid function with all types tendering bids that are $2/3$ of their valuation. When m_n bidders enter the auction, the presence of the m_n bidders means a group 1 bidder should behave more aggressively (or at least no less competitively) given the presence of the same $(m_1 - 1)$ rivals and the m_n bidders. This behavior propagates through to the group 1 bidders with types that exceed the highest group n type given they now face more aggressive behavior from their rivals with lower valuations. The symmetric auction inverse bid function involving $m_1 = 3$ bidders is depicted as the lighter, dashed line which we have labeled $\varphi_1^s(b; m_1 = 3)$. More aggressive behavior would mean that the $\varphi_1(b)$ function in

the asymmetric auction should lie between this linear function and the 45° line. In fact, the exact opposite is happening in the figure as bidders shade their valuations by even more.¹⁴ Were one to assume this was the equilibrium as opposed to the (correct) one in the panel on the left involving bid-separation, any inference based off these approximations would suggest drastically different implications. For example, consider ranking the first-price auction relative to a second-price auction according to expected revenues. The correct solution suggests slightly higher revenues can be expected from a first-price auction, while the incorrect solution predicts expected revenues from a first-price auction that are less than half of the amount that can be expected from a second-price auction.¹⁵ Thus, the incorrect solution not only substantially miscalculates expected revenue, but it would lead to the wrong revenue ranking as well.

EXAMPLE 2: Consider an auction with $m_1 = 3$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 0.95]$. In this setting, $b^c = 0.85$. If the m_1 bidders were at a symmetric auction, the equilibrium strategy would again prescribe bidders tender $2/3$ of their valuations since their distribution did not change from Example 1. Importantly, because now $\bar{b}^s < b^c$, the condition of Corollary 2 is not met (again, nor is the Corollary 1 condition) meaning bidders will tender offers over a common support. Moreover, applying what we know from Proposition 2(a): $\bar{v}_1/\bar{v}_n = 1.0526 < 1.0811 = \tau$ so the necessary and sufficient condition which applies to this example confirms that bidding should happen on the same support for all types. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 6(a).

This time the algorithm that imposes (appropriately now) the boundary conditions for a common bid support succeeds in solving for the correct bid function. Building on the discussion above involving how behavior should compare across a m_1 -symmetric

¹⁴Though we depict this improper approximation, we should note that if the wrong boundary conditions are imposed, the nonlinear optimization solver we use (SNOPT) cannot achieve convergence and typically reports that it cannot proceed into an undefined region. We are reassured by this as it suggests the problem is sufficiently well disciplined and users would recognize immediately any difficulties.

¹⁵Expected revenue from a second-price auction is 0.5949. Using the correct bid-separation solution, the winning bid (tendered by a group 1 bidder) exceeds \bar{b}_n 47% of the time and expected revenue increases slightly to 0.5959. In contrast, expected revenue from the incorrect, common bid support approximation is 0.2539.

auction and a (m_1, m_n) -asymmetric auction, the inverse bidding strategies suggest all bidders are now more aggressive in equilibrium. In this example the bidders are nearly identical. If all bidders were group 1 bidders, then the symmetric auction equilibrium strategy would involve bidders tendering $4/5$ of their valuation. In this asymmetric setting, since group n bidders are slightly weaker than group 1 bidders, we see the common high bid is just below $4/5$ and there is only a slight separation in behavior at the upper end of the bid support. Note, too, that the solution to this problem, even though bid-separation does not obtain, was previously unattainable by researchers solving asymmetric auctions since $m_1 \geq 2$ and the type supports are different. Rather than present another approximation using the wrong boundary conditions, we note that the discussion provided above about violating equilibrium bidding behavior when m_n bidders enter the auction holds in this setting too; that is, improperly imposing bid-separation fails to converge to a solution and in searching for a solution continues to stray into territory which we know is theoretically invalid because m_1 bidders shade their valuations more.

EXAMPLE 3: As a final example based on bidders who receive valuations from uniform distributions, consider an auction with $m_1 = 2$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 0.15]$. The extreme difference in the highest possible valuations of bidders might best correspond with the motivating example we provided in the introduction which involved two billionaire art collectors vying for a painting at an auction with two art students. We are likely being quite generous in evaluating the competitiveness of the art students in this example—if the highest possible valuation for the billionaires is just \$1,000,000, the highest possible valuation of students is still \$150,000. Regardless, in this setting, bid-separation is certain—the sufficient conditions of Corollary 1 and 2 as well as the necessary and sufficient condition of Proposition 2(b) for bid-separation are easily satisfied. Assuming a common bid support in this setting would be disastrous as, in equilibrium, less than 18% of the strong types tender bids that fall within the bid support of the weak bidders. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 6(b). If a common bid support were assumed, then no bid could exceed 0.15 which assumes students bid as aggressively as possible by tendering their valuations. Actually, the high bid for the strong group (0.51) is more than three times the highest type of the weak group.

Since a symmetric auction with only the two billionaires would involve each tendering half of their valuation in equilibrium, the mere presence of the student bidders leads to an increase in the bids of all strong types which propagates through to the (much larger) share of the strong group's bid support for which the weaker bidders are not even active.

EXAMPLE 4: Having given some credibility to the solution approach suggested by considering examples with uniform distributions, let us venture into territory that is less convenient analytically. The general solution approach we provide is not limited in any way by the distribution considered so long as the standard conditions for uniqueness are satisfied (see the conditions we discussed at the beginning of Section 2, just before Lemma 1). Consider instead an auction with two bidders from each group at auction. Let the valuations of group 1 bidders be distributed according to a Normal distribution with mean 0.5 and standard deviation 0.25 which has been truncated over the support $[0, 1]$. Likewise, assume group n bidders draw types which are distributed according to a Normal distribution with mean $\bar{v}_n/2$ and standard deviation 0.25, which has been truncated over the support $[0, 0.75]$. In this example, the type distribution of group 1 bidders dominates that of group n bidders in terms of the reverse hazard rate. Theory tells us this stochastic ranking of distributions is a sufficient condition for weakness to breed aggression on overlapping portions of the bid supports; for a given type, group n bidders should tender an offer that exceeds that of group 1 bidders. For this example, $b^c = 1/2$ and step 4 of the solution procedure suggests bid-separation occurs. Indeed, our approximated solutions, depicted in Figure 7(a) reflect this feature of the solution. Note, too, that the inverse bid function of group 1 bidders is more nonlinear than the solutions involving uniform distributions but, nonetheless, the solution is still smooth at (\bar{b}_n, \hat{v}) . Lastly, note that if \bar{v}_n is increased from 0.75 to 0.82 (or above) the equilibrium involves all bidders tendering offers over a common bid support. We continue this thought in the next example.

EXAMPLE 5: Again let valuations of group 1 bidders be distributed according to a Normal distribution with mean 0.5 and standard deviation 0.25 which has been truncated over the support $[0, 1]$. Now, let's increase the highest type of group n bidders to 0.85. Assume group n bidders draw types which are distributed according to a Normal distribution with mean $\bar{v}_n/2$ and standard deviation 0.25, which has been truncated over the support $[0, 0.85]$. At the end of the last example we noted that if

$(m_1, m_n) = (2, 2)$, bid-separation would not occur. This implication is in the spirit of the first part of Corollary 3 (which applied to the uniform model from Proposition 2): the weak group becomes stronger because their highest possible type increases (as does the mean of their valuation distribution which is tied to the highest type by construction) and so bid-separation is less likely. To revisit the implications of the second parts of Corollary 3 and 4 in an situation outside of the uniform model, suppose another bidder shows up at auction from each group so that $(m_1, m_n) = (3, 3)$. Step 4 of the general solution procedure suggests bid-separation occurs, consistent with the implications of Corollary 3 and 4, though in a setting for which they are not known to hold. We depict the equilibrium bid functions in Figure 7(b). Again, because reverse hazard rate dominance holds, weakness leads to aggression over the common part of the bid support. Perhaps not surprisingly, because participation has increased by 50%, all bidders behave more competitively in tendering higher bids. Bid shading decreases as does the difference in the equilibrium bid functions of the two groups so that the only real separation in behavior occurs as types increase towards \hat{v} . Importantly, this example serves as a reminder that bid-separation depends on both the relative difference between supports (\bar{v}_1/\bar{v}_n) as well as the composition of (and number of) bidders at auction (m_1, m_n) . This can be important when considering any number of counterfactual simulations (perhaps after having estimated type distributions in empirical work) such as the effect of preference policies and bidder subsidies, particularly when entry is endogenous so that participation may vary with the policy considered.

5 Conclusion

This paper focuses on an empirically relevant complication that arises in first-price auctions with more than two bidders, namely bid-separation. Our theoretical and numerical results suggest that bid-separation may occur in the face of even relatively small asymmetries. Nevertheless, most existing empirical and numerical methods do not take bid-separation into account. A main motivation of the paper is to make applied researchers aware of this oversight and to take some initial steps toward satisfactorily incorporating considerations of bid-separation into both theory and practice.

It should be noted that bid-separation may well exist outside the particular setting we have investigated. We have, for simplicity, concentrated on first-price auctions with

exactly two groups of bidders. Allowing for more groups of bidders does not appear to make bid-separation any less likely. Similarly, other pay-your-bid auctions—most notably the all-pay auction—will be prone to bid-separation as well.

Even with an arbitrary number of groups of bidders, Lemma 1 provides some equilibrium structure. We speculate that a “recursive” version of Proposition 1 could be developed to make inferences about the maximum bids of different groups and the critical types that submit bids that coincide with the maximum bid of some other group. The details, however, are left for future research.

In many ways, the all-pay auction is analytically even more complicated than the first-price auction. Parreiras and Rubinchik (2010) prove that bidding strategies need not even be continuous when there are more than two asymmetric bidders. In a related paper, Kirkegaard (2013) proves that a bidder may become worse off if he is a member of a diverse set of bidders who are given preferential treatment in an all-pay auction with more than two bidders. bid-separation simply adds to these complications. We have elected to focus on the simpler first-price auction, though this choice is primarily motivated by the empirical relevance of the auction format.

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Appendix A: Proofs

Proof of Lemma 1. As mentioned, the result follows from Lebrun (2006). Here, we prove the result by extending Kirkegaard’s (2009, footnote 12) “revealed preference” approach to allow $\bar{v}_i \geq \bar{v}_j$. Thus, by contradiction, assume $\bar{v}_i \geq \bar{v}_j$ but $\bar{b}_i < \bar{b}_j$. Let $P(b)$ be the distribution function of the highest bid among bidder i ’s and bidder j ’s common rivals, with $P(\bar{b}_j) \geq P(\bar{b}_i)$. Consider bidder j with type \bar{v}_j . In order for \bar{b}_j to an equilibrium bid, there must be no incentive to deviate to \bar{b}_i instead. For either bid, bidder j outbids bidder i with probability 1, so the requirement is that

$$(\bar{v}_j - \bar{b}_j) P(\bar{b}_j) \geq (\bar{v}_j - \bar{b}_i) P(\bar{b}_i)$$

or

$$\bar{v}_j [P(\bar{b}_j) - P(\bar{b}_i)] \geq \bar{b}_j P(\bar{b}_j) - \bar{b}_i P(\bar{b}_i). \quad (10)$$

Similarly, bidder i with type \bar{v}_i must weakly prefer bidding \bar{b}_i to \bar{b}_j . Since a bid of \bar{b}_i causes bidder i to lose to bidder j with strictly positive probability, payoff from bidding \bar{b}_i is strictly smaller than $(\bar{v}_i - \bar{b}_i) P(\bar{b}_i)$. In comparison, deviating to \bar{b}_j leads bidder i to outbid bidder j with probability 1. Thus, it is necessary that

$$(\bar{v}_i - \bar{b}_i) P(\bar{b}_i) > (\bar{v}_i - \bar{b}_j) P(\bar{b}_j),$$

or

$$\bar{b}_j P(\bar{b}_j) - \bar{b}_i P(\bar{b}_i) > \bar{v}_i [P(\bar{b}_j) - P(\bar{b}_i)].$$

Since the term in the brackets is non-negative and $\bar{v}_i \geq \bar{v}_j$, the last inequality implies that

$$\bar{b}_j P(\bar{b}_j) - \bar{b}_i P(\bar{b}_i) > \bar{v}_j [P(\bar{b}_j) - P(\bar{b}_i)]. \quad (11)$$

The proof concludes by observing that (10) and (11) are contradictory. ■

Proof of Lemma 2. Assume $\bar{b}_n < \bar{b}_1$. Consider first bidder n with type \bar{v}_n . Let $U_n(b)$ denote the natural logarithm of his expected payoff from bidding b , with

$$U'_n(b) = \frac{-1}{\bar{v}_n - b} + (m_n - 1) \frac{d}{db} \ln F_n(\varphi_n(b)) + m_1 \frac{d}{db} \ln F_1(\varphi_1(b)) \text{ for } b \in (\underline{v}, \bar{b}_n)$$

and

$$U'_n(b) = \frac{-1}{\bar{v}_n - b} + m_1 \frac{d}{db} \ln F_1(\varphi_1(b)) \text{ for } b \in (\bar{b}_n, \bar{b}_1).$$

Let $U'_n(\bar{b}_n \nearrow)$ and $U'_n(\bar{b}_n \searrow)$ denote the left-derivative and right-derivative at \bar{b}_n , respectively. In order for bidder n to not submit higher bids than \bar{b}_n it is necessary that $U'_n(\bar{b}_n \searrow) \leq 0$. For similar reasons, it is necessary that $U'_n(\bar{b}_n \nearrow) \geq 0$. However, $U'_n(\bar{b}_n \nearrow) > 0$ can be ruled out, because in this case types marginally below \bar{b}_n would find it profitable to deviate from their equilibrium bid and instead bid \bar{b}_n . Thus, $U'_n(\bar{b}_n \nearrow) = 0$. Combining these observations yields

$$\frac{d}{db} \ln F_n(\varphi_n(\bar{b}_n \nearrow)) \geq \frac{m_1}{m_n - 1} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \searrow)) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \nearrow)) \right]. \quad (12)$$

Consider next bidder 1 with type $\hat{v} \in (v, \bar{b}_1)$. Let $U_1(b)$ denote the natural logarithm of his expected payoff from bidding b . The derivative, $U'_1(b)$, can be calculated in much the same manner as $U'_n(b)$, the main difference being that the composition of rival bidders is different. In equilibrium, this bidder is supposed to find a bid of \bar{b}_n optimal. By the above arguments, it is thus necessary that $U'_1(\bar{b}_n \nearrow) = U'_1(\bar{b}_n \searrow) = 0$, which implies that

$$\frac{d}{db} \ln F_n(\varphi_n(\bar{b}_n \nearrow)) = \frac{m_1 - 1}{m_n} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \searrow)) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \nearrow)) \right]. \quad (13)$$

Since φ_n is non-decreasing, the term in brackets must be non-negative. If it is strictly positive, then (12) implies

$$\begin{aligned} \frac{d}{db} \ln F_n(\varphi_n(\bar{b}_n \nearrow)) &\geq \frac{m_1}{m_n - 1} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \searrow)) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \nearrow)) \right] \\ &> \frac{m_1 - 1}{m_n} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \searrow)) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_n \nearrow)) \right], \end{aligned}$$

which contradicts (13). Thus, the bracketed term must be zero, and it then follows immediately from (13) that the left-derivative of φ_n at \bar{b}_n , which we for simplicity denote $\varphi'_n(\bar{b}_n)$, must be zero as well (incidentally, it also follows that φ_1 does not have a kink at \bar{b}_n). ■

Proof of Corollary 2. To prove the corollary it is sufficient to establish that $\bar{b}_1 > \bar{b}_1^s$. Let $EU_1^s(v)$ denote bidder 1's expected utility, as a function of his type, in

a symmetric auction against $m_1 - 1 \geq 1$ identical rivals. Myerson (1981) has shown that

$$EU_1^s(v) = EU_1^s(\underline{v}_1) + \int_{\underline{v}_1}^v q_1^s(x) dx,$$

where $q_1^s(x) = F_1(x)^{m_1-1}$ is the probability of winning for a bidder with type x . Of course, it must also hold that $EU_1^s(v) = (v - b_1^s(v))q_1^s(v)$, where $b_1^s(v)$ is the bidding strategy in the symmetric auction. Since $m_1 \geq 2$, it holds that $EU_1^s(\underline{v}_1) = 0$, since a type \underline{v}_1 bidder wins with probability zero. Hence, since $q_1^s(\bar{v}_1) = 1$,

$$\bar{v}_1 - \bar{b}_1^s = \int_{\underline{v}_1}^{\bar{v}_1} F_1(x)^{m_1-1} dx.$$

In the asymmetric auction we have

$$\bar{v}_1 - \bar{b}_1 = \int_{\underline{v}_1}^{\bar{v}_1} F_n(\varphi_n(b_1(x)))^{m_n} F_1(x)^{m_1-1} dx < \int_{\underline{v}_1}^{\bar{v}_1} F_1(x)^{m_1-1} dx = \bar{v}_1 - \bar{b}_1^s,$$

where $b_1(x)$ is bidder 1's equilibrium bidding strategy in the asymmetric auction. It follows that $\bar{b}_1 > \bar{b}_1^s$. ■

Proof of Proposition 2. Fix m_1 and consider a change in m_n . The proposition is trivially true if $\hat{v} = \bar{v}_1$. Hence, assume $\hat{v} < \bar{v}_1$. Since the relationship in Proposition 1 does not depend on m_n , it follows that $\hat{v}' < \hat{v}$ if and only if $\bar{b}'_n > \bar{b}_n$, where \bar{b}'_n and \bar{b}_n are the maximum bids of bidder n in the environment with (m_1, m'_n) and (m_1, m_n) bidders, respectively. Thus, assume by contradiction that $\bar{b}'_n \leq \bar{b}_n$.

The proof proceeds in two steps. First, inverse bidding strategies in the two environments are compared. In the second step, this comparison makes it possible to contradict the starting assumption that $\bar{b}'_n \leq \bar{b}_n$.

Consider first the case where the inequality is strict, or $\bar{b}'_n < \bar{b}_n$ and thus $\hat{v}' > \hat{v}$. Let $\gamma_i(b)$ denote the inverse bidding strategies with (m_1, m'_n) bidders and let $\varphi_i(b)$ denote the inverse bidding strategies with (m_1, m_n) bidders. By assumption, $\gamma_i(\bar{b}'_n) > \varphi_i(\bar{b}'_n)$, $i = 1, n$. Now reduce the bid from \bar{b}'_n until the first point is reached (if one exists) for which $\gamma_i(b) = \varphi_i(b)$ but $\gamma_j(b) \geq \varphi_j(b)$, for some $i = 1, n$, with $i \neq j$. If

$i = n$, then from (2)

$$\begin{aligned} \frac{d}{db} \ln F_n(\gamma_n(b)) &= \frac{1}{m_1 + m'_n - 1} \left[\frac{m_1}{\gamma_1(b) - b} - \frac{m_1 - 1}{\gamma_n(b) - b} \right] \\ &< \frac{1}{m_1 + m_n - 1} \left[\frac{m_1}{\varphi_1(b) - b} - \frac{m_1 - 1}{\varphi_n(b) - b} \right] \\ &= \frac{d}{db} \ln F_n(\varphi_n(b)). \end{aligned}$$

However, this contradicts the fact that $\gamma_n > \varphi_n$ to the right of b (as b is the highest bid at which γ_n and φ_n intersects and $\gamma_n(\bar{b}'_n) > \varphi_n(\bar{b}'_n)$). Thus, assume instead that $\gamma_1(b) = \varphi_1(b)$ but $\gamma_n(b) \geq \varphi_n(b)$. Here, we wish to compare $\frac{d}{db} \ln F_1(\gamma_1(b))$ and $\frac{d}{db} \ln F_1(\varphi_1(b))$. If the former is strictly smaller than the latter, we obtain the same contradiction as above. On the other hand, it is easy to verify that

$$\frac{d}{db} \ln F_1(\gamma_1(b)) \geq \frac{d}{db} \ln F_1(\varphi_1(b))$$

at some b where $\gamma_1(b) = \varphi_1(b)$ but $\gamma_n(b) \geq \varphi_n(b)$ implies that

$$\frac{m_1}{\varphi_1(b) - b} - \frac{m_1 - 1}{\varphi_n(b) - b} \leq 0,$$

which in turn means that $\frac{d}{db} \ln F_1(\varphi_1(b)) \leq 0$. However, this contradicts the equilibrium property that $\frac{d}{db} \ln F_1(\varphi_1(b)) > 0$ in the interior. In other words, there can be no intersection between γ_i and φ_i as b is reduced from \bar{b}'_n . Thus, we conclude that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \bar{b}'_n]$, $i = 1, n$.

The second case is $\bar{b}'_n = \bar{b}_n$ and $\hat{v}' = \hat{v}$. Here, it can be shown that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \bar{b}'_n)$. A sketch is given next (with details available on request). By assumption, $\gamma_i(\bar{b}'_n) = \varphi_i(\bar{b}'_n)$. Moreover,

$$\begin{aligned} \frac{d}{db} \ln F_n(\gamma_n(b))|_{b=\bar{b}'_n} &= \frac{d}{db} \ln F_n(\varphi_n(b))|_{b=\bar{b}'_n} = 0 \\ \frac{d}{db} \ln F_1(\gamma_1(b))|_{b=\bar{b}'_n} &= \frac{d}{db} \ln F_1(\varphi_1(b))|_{b=\bar{b}'_n} = \frac{1}{m_1} \frac{1}{\bar{v}_n - \bar{b}'_n}. \end{aligned}$$

However, simple differentiation and tedious algebra can be used to prove that

$$\frac{d^2}{db^2} \ln F_1(\gamma_1(b))|_{b=\bar{b}'_n} > \frac{d^2}{db^2} \ln F_1(\varphi_1(b))|_{b=\bar{b}'_n}$$

and thus

$$\frac{d}{db} \ln F_1(\gamma_1(b)) < \frac{d}{db} \ln F_1(\varphi_1(b))$$

for b close to, but strictly below, \bar{b}'_n . In other words, $\gamma_1(b) > \varphi_1(b)$ for b close to, but strictly below, \bar{b}'_n . This property can then be used to establish that $\gamma_n(b) > \varphi_n(b)$ for b close to, but strictly below, \bar{b}'_n . The argument from the first case ($\bar{b}'_n = \bar{b}_n$) then applies to prove that $\gamma_i(b) > \varphi_i(b)$, for all $b \in (r, \bar{b}'_n)$.

The next step utilizes (1). Specifically, the above ranking of inverse bidding strategies implies that

$$\begin{aligned} \frac{d}{db} \ln F_1(\gamma_1(b))^{m_1-1} F_n(\gamma_n(b))^{m'_n} &= \frac{1}{\gamma_1(b) - b} \\ &< \frac{1}{\varphi_1(b) - b} \\ &= \frac{d}{db} \ln F_1(\varphi_1(b))^{m_1-1} F_n(\varphi_n(b))^{m_n} \end{aligned}$$

for all $b \in (r, \bar{b}'_n)$. Equivalently

$$\frac{d}{db} \left[\ln \left(\frac{F_1(\gamma_1(b))}{F_1(v')} \right)^{m_1-1} F_n(\gamma_n(b))^{m'_n} \right] < \frac{d}{db} \left[\ln \left(\frac{F_1(\varphi_1(b))}{F_1(\varphi_1(\bar{b}'_n))} \right)^{m_1-1} \left(\frac{F_n(\varphi_n(b))}{F_n(\varphi_n(\bar{b}'_n))} \right)^{m_n} \right].$$

The two terms in brackets coincide at $b = \bar{b}'_n$, where they are both equal to zero. Since $r > v$, both bracketed terms converge to finite values as $b \rightarrow r$. However, since the bracketed term on the left is flatter than its counterpart on the right, it must hold that

$$\ln \left(\frac{F_1(r)}{F_1(v')} \right)^{m_1-1} F_n(r)^{m'_n} > \ln \left(\frac{F_1(r)}{F_1(\varphi_1(\bar{b}'_n))} \right)^{m_1-1} \left(\frac{F_n(r)}{F_n(\varphi_n(\bar{b}'_n))} \right)^{m_n}$$

since $\gamma_i(r) = \varphi_i(r) = r$, $i = 1, n$. Since $v' = \gamma_1(\bar{b}'_n) \geq \varphi_1(\bar{b}'_n)$ and $F_n(\varphi_n(\bar{b}'_n)) \leq 1$,

$$\left(\frac{F_1(r)}{F_1(v')} \right)^{m_1-1} F_n(r)^{m'_n} > \left(\frac{F_1(r)}{F_1(v')} \right)^{m_1-1} (F_n(r))^{m_n}$$

or $F_n(r)^{m'_n} > (F_n(r))^{m_n}$. However, since $F_n(r) \in (0, 1)$ and $m'_n > m_n$, this is impossible. Hence, a contradiction to the assumption that $\bar{b}'_n \leq \bar{b}_n$ has now been obtained.

Next, fix m_n instead and let m_1 increase to m'_1 . Again, the proposition is trivially true if $\hat{v} = \bar{v}_1$. Hence, assume $\hat{v} < \bar{v}_1$. Note that the relationship in Proposition 1 depends on m_1 . The downwards sloping line in Figure 1 pivots counterclockwise around the point (\bar{v}_w, \bar{v}_w) , such that b^c moves to the left. Thus, if $\hat{v}' \geq \hat{v}$ then $\bar{b}'_n < \bar{b}_n$ is necessary. Assume by contradiction that $\hat{v}' \geq \hat{v}$ and note now that $\gamma_1(\bar{b}'_n) = \hat{v}' > \hat{v} = \varphi_1(\bar{b}_n) > \varphi_1(\bar{b}'_n)$ and likewise that $\gamma_n(\bar{b}'_n) = \bar{v}_n = \varphi_n(\bar{b}_n) > \varphi_n(\bar{b}'_n)$. Hence, $\gamma_i(\bar{b}'_n) > \varphi_i(\bar{b}'_n)$, $i = 1, n$. Following the same steps as above then yields the contradiction.

Thus, it has now been established that if either m_1 or m_n increases then $\hat{v}' \leq \hat{v}$ with $\hat{v}' < \hat{v}$ if $\hat{v} < \bar{v}_1$. Since increases in m_1 and m_n move the equilibrium in the same direction, the same conclusion holds if both m_1 and m_n increase at the same time. This concludes the proof. ■

Proof of Proposition 3. Let bidder i 's inverse bidding strategy be denoted $\varphi_i^F(b)$ and $\varphi_i^G(b)$ in the two scenarios where distributions are (F_1, F_n) and (G_1, G_n) , respectively (recall the assumption that equilibrium is unique). The case where $G_1 = F_1$ and $G_n = F_n$ is uninteresting. Thus, assume in the remainder that $G_1 \succ F_1$ and/or $G_n \succ F_n$. The system in (2) can be written as

$$\varphi'_i(b) = \frac{1}{n-1} \frac{F_i(\varphi_i(b))}{f_i(\varphi_i(b))} \left[\frac{m_j}{\varphi_j(b) - b} - \frac{m_j - 1}{\varphi_i(b) - b} \right] \quad (14)$$

where $j \neq i$, $i = 1, 2$.

Assume by contradiction that $\bar{b}_n^G < \bar{b}_n^F$, which by Proposition 1 implies $\hat{v}^G \geq \hat{v}^F$. Hence, $\varphi_i^G(\bar{b}_n^G) > \varphi_i^F(\bar{b}_n^G)$, $i = 1, 2$. Now move leftwards (reducing b) from \bar{b}_n^G . Let $b' > r$ denote the first (i.e. highest) bid for which $\varphi_i^G(b') = \varphi_i^F(b')$ for some (or both) i , if it exists. If it exists, there are two possibilities. One possibility is that the crossing occurs at the same place for both $i = 1$ and $i = 2$, i.e. $\varphi_1^G(b') = \varphi_1^F(b')$ and $\varphi_2^G(b') = \varphi_2^F(b')$. The bracketed term in (14) is then the same for both scenarios. However, since $G_i \succ F_i$ for some i , it follows that $\varphi_i^G(b') < \varphi_i^F(b')$. However, this contradicts the fact that $\varphi_i^G(b') = \varphi_i^F(b')$ to the right of b' . The other possibility is that $\varphi_i^G(b') = \varphi_i^F(b')$ but $\varphi_j^G(b') > \varphi_j^F(b')$, $j \neq i$. The conclusion is again that $\varphi_i^G(b') < \varphi_i^F(b')$, because the bracketed term in (14) is smaller (and the first term is no larger) when distributions are (G_1, G_n) compared to (F_1, F_n) . The same contradiction is thus achieved. It now follows that $\varphi_i^G(b) > \varphi_i^F(b)$ for all $b \in (r, \bar{b}_n^G]$, $i = 1, 2$.

Assuming that $G_1 \succ F_1$ (the proof is similar if $G_n \succ F_n$ instead) it follows from (1) that

$$\begin{aligned} \frac{d}{db} \ln F_n(\varphi_n^F(b))^{m_n} F_1(\varphi_1^F(b))^{m_1-1} &= \frac{1}{\varphi_1^F(b) - b} \\ &> \frac{1}{\varphi_1^G(b) - b} \\ &= \frac{d}{db} \ln G_n(\varphi_n^G(b))^{m_n} G_1(\varphi_1^G(b))^{m_1-1}, \end{aligned}$$

or

$$\frac{d}{db} \left[\ln \left(\frac{F_n(\varphi_n^F(b))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(\varphi_1^F(b))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} \right] > \frac{d}{db} \left[\ln \left(\frac{G_n(\varphi_n^G(b))}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(b))}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1} \right]$$

for all $b \in (r, \bar{b}_n^G]$. Evaluated at $b = \bar{b}_n^G$, the bracketed term on either side of the inequality are both zero. Since $r > \underline{v}$, the bracketed term on the left converges to a finite value as $b \rightarrow r$. Moreover, since the bracketed term on the left is steeper in b than the bracketed term on the right, the latter must also converge to a finite value, with

$$\ln \left(\frac{F_n(\varphi_n^F(r))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(\varphi_1^F(r))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \ln \left(\frac{G_n(\varphi_n^G(r))}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(r))}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1}$$

Since $(\varphi_1^F, \varphi_n^F)$ are equilibrium strategies, it must hold that $\varphi_i^F(r) = r$, $i = 1, 2$. If $(\varphi_1^G, \varphi_n^G)$ are equilibrium strategies as well, then it also holds that $\varphi_i^G(r) = r$, $i = 1, 2$, and we have

$$\left(\frac{F_n(r)}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(r)}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(r)}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1}$$

or

$$\left(\frac{G_n(\varphi_n^G(\bar{b}_n^G))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(\bar{b}_n^G))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{F_n(r)} \right)^{m_n} \left(\frac{G_1(r)}{F_1(r)} \right)^{m_1-1}$$

which, since $\varphi_i^G(\bar{b}_n^G) > \varphi_i^F(\bar{b}_n^G)$, implies

$$\left(\frac{G_n(\varphi_n^G(\bar{b}_n^G))}{F_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(\bar{b}_n^G))}{F_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{F_n(r)} \right)^{m_n} \left(\frac{G_1(r)}{F_1(r)} \right)^{m_1-1}$$

However, the assumption that $G_i \succ F_i$ ($G_i = F_i$) is equivalent to $\frac{d}{dv} \frac{G(v)}{F(v)} > 0$ ($\frac{d}{dv} \frac{G(v)}{F(v)} = 0$). Consequently, the above inequality must be violated. In other words, $(\varphi_1^G, \varphi_n^G)$ cannot form an equilibrium. ■

Proof of Proposition 5. The relationship in Proposition 1 characterizes a necessary condition on any candidate (\hat{v}, \bar{b}_n) pair. The next step is to use mechanism design arguments to derive a second necessary condition. The final step combines these two conditions to establish Proposition 5.

As in any mechanism design argument, the equilibrium allocation plays an important role. Thus, let $q_i(v)$ denote the probability that a bidder in group i , $i = 1, n$, wins the auction if his type is v . Letting $EU_i(v)$ denote such a bidder's expected utility, Myerson (1981) has shown that

$$EU_i(v) = EU_i(\underline{v}_i) + \int_{\underline{v}_i}^v q_i(x) dx.$$

In the setting in Proposition 5, it is easily seen that $EU_i(\underline{v}_i) = 0$ (recall that $\underline{v}_i = 0$). Consider now the highest types, \bar{v}_1 and \bar{v}_n , respectively. First, observe that

$$\begin{aligned} EU_1(\bar{v}_1) &= EU_1(\hat{v}) + \int_{\hat{v}}^{\bar{v}_1} q_1(x) dx \\ &= (\hat{v} - \bar{b}_n) \left(\frac{\hat{v}}{\bar{v}_1} \right)^{m_1-1} + \int_{\hat{v}}^{\bar{v}_1} \left(\frac{x}{\bar{v}_1} \right)^{m_1-1} dx, \end{aligned}$$

since type $x \geq \hat{v}$ outbids all group n bidders with probability one and thus wins if all rival bidders in group i have types that are below x . Conveniently, this expression does not require any knowledge of $q_1(x)$ for $x < \hat{v}$. Integrating now yields the conclusion that

$$\int_{\underline{v}_1}^{\bar{v}_1} q_1(x) dx = (\hat{v} - \bar{b}_n) \left(\frac{\hat{v}}{\bar{v}_1} \right)^{m_1-1} + \frac{1}{m_1} \frac{\bar{v}_1^{m_1} - \hat{v}^{m_1}}{\bar{v}_1^{m_1-1}}. \quad (15)$$

Similarly, since

$$EU_n(\bar{v}_n) = (\bar{v}_n - \bar{b}_n) \left(\frac{\hat{v}}{\bar{v}_1} \right)^{m_1},$$

it follows that

$$\int_{\underline{v}_n}^{\bar{v}_n} q_n(x) dx = (\bar{v}_n - \bar{b}_n) \left(\frac{\hat{v}}{\bar{v}_1} \right)^{m_1}. \quad (16)$$

The ex ante probability that any given bidder wins the auction takes a particularly

useful form when distributions are uniform, since

$$\int_{\underline{v}_i}^{\bar{v}_i} q_i(x) f_i(x) dx = \frac{1}{\bar{v}_i} \int_{\underline{v}_i}^{\bar{v}_i} q_i(x) dx.$$

Since the auction has no reserve price, the item will be sold for sure. In other words, the ex ante winning probabilities must aggregate to one, or

$$m_1 \frac{1}{\bar{v}_1} \int_{\underline{v}_1}^{\bar{v}_1} q_1(x) dx + m_n \frac{1}{\bar{v}_n} \int_{\underline{v}_n}^{\bar{v}_n} q_n(x) dx = 1. \quad (17)$$

Combining (15) and (16) with (17) yields the necessary condition that

$$\bar{b}_n = \frac{m}{\bar{b}_n m_1 + m_n \hat{v}} \bar{v}_n \hat{v} \quad (18)$$

for $\hat{v} \in [0, \bar{v}_1]$, or, stated differently,

$$\hat{v} = \frac{\bar{v}_n m_1 \bar{b}_n}{m_n (\bar{v}_n - \bar{b}_n) + \bar{v}_n (m_1 - 1)} \quad (19)$$

with the restriction that \bar{b}_n is such that $\hat{v} \in [0, \bar{v}_1]$.

In summary, any equilibrium (\hat{v}, \bar{b}_n) pair must satisfy both (19) and (7). Thus, the next step is to characterize what turns out to be the unique (\hat{v}, \bar{b}_n) pair that satisfies both conditions. First, note that the right hand side of (19) is strictly increasing in \bar{b}_n and ranges from 0 to $\frac{m_1}{m_1-1} \bar{v}_n$ as \bar{b}_n increases from 0 to \bar{v}_n . However, the term $\frac{m_1}{m_1-1} \bar{v}_n - \frac{1}{m_1-1} \bar{b}_n$ on the right hand side of (7) is strictly decreasing in \bar{b}_n and ranges from $\frac{m_1}{m_1-1} \bar{v}_n$ to \bar{v}_n as \bar{b}_n increases from 0 to \bar{v}_n . Thus, the two equations (i.e. (19) and $\hat{v} = \frac{m_1}{m_1-1} \bar{v}_n - \frac{1}{m_1-1} \bar{b}_n$) must have a unique intersection with $\bar{b}_n \in (0, \bar{v}_n)$. We first identify this intersection and then subsequently check whether it satisfies the feasibility condition that $\hat{v} \in [0, \bar{v}_1]$. Equalizing these two equations yields a quadratic equation in \bar{b}_n . The larger root can be ruled out, since it yields the conclusion that $\bar{b}_n > \bar{v}_n$. The smaller root is $\bar{b}_n = \kappa(m_1, m_n) \bar{v}_n$, for which $\hat{v} = \tau(m_1, m_n) \bar{v}_n$. This candidate satisfies the final feasibility condition that $\hat{v} \leq \bar{v}_1$ if and only if $\tau(m_1, m_n) \leq \frac{\bar{v}_1}{\bar{v}_n}$. This proves the second part of the proposition. If $\tau(m_1, m_n) > \frac{\bar{v}_1}{\bar{v}_n}$, the condition that $\hat{v} \leq \bar{v}_1$ instead binds. Nevertheless, (19) and (18) must be satisfied. The latter establishes the characterization in the first part of the proposition.

Continuity follows from the continuity of (19) and (7). Of course this implies that

when $\tau(m_1, m_n)$ is identical to $\frac{\underline{v}_1}{\underline{v}_n}$, the equilibrium pair (\widehat{v}, \bar{b}_n) in the two parts of the proposition coincide. ■

Appendix B: Initial Conditions

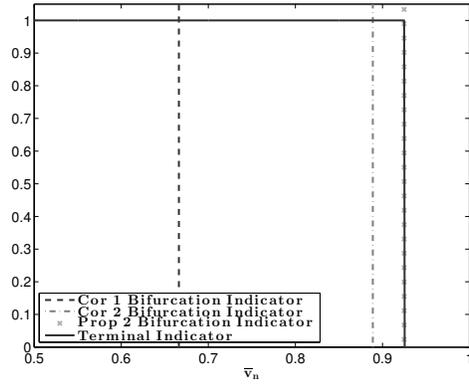
Initial conditions are derived from the bidding behavior of bidders with low types. Imagine first that $\underline{v}_1 > \underline{v}_n$. Since $m_1 \geq 2$, competition among the first group of bidders ensure that they will bid at least \underline{v}_1 . Thus, bidder n wins with probability zero if his type is below \underline{v}_1 . Similarly, if $m_n \geq 2$ and $\underline{v}_n > \underline{v}_1$ then bidders in the first group stand no chance of winning if their types are below \underline{v}_n . Stated differently, in both cases any bidder has a strictly positive chance of winning if and only if his type strictly exceeds $\underline{v} = \max\{\underline{v}_1, \underline{v}_n\}$. This insight provides the “initial condition” that $\varphi_i(\underline{v}) = \underline{v}$, $i = 1, n$.

However, the above discussion does not include the possibility that $m_n = 1$ and $\underline{v}_n > \underline{v}_1$. In this case, it is no longer true that bidder n of type $\underline{v} = \underline{v}_n$ bids his true value. There is an incentive to bid lower, as he may still win in the event that his rivals all have types below \underline{v} . Following Maskin and Riley (2000) and Lebrun (2006), the initial condition can nevertheless be uniquely and explicitly characterized. Specifically, among the first group of bidders there is a threshold value, $\underline{b} \in (\underline{v}_1, \underline{v}_n)$, such that any bidder bids his true value (and wins with probability zero) if his type is below \underline{b} . In equilibrium, $\varphi_n(\underline{b}) = \underline{v}_n$. Hence, bids above \underline{b} are what Lebrun (2006) terms “serious bids” as they entail a strictly positive winning probability. The equilibrium value of \underline{b} is

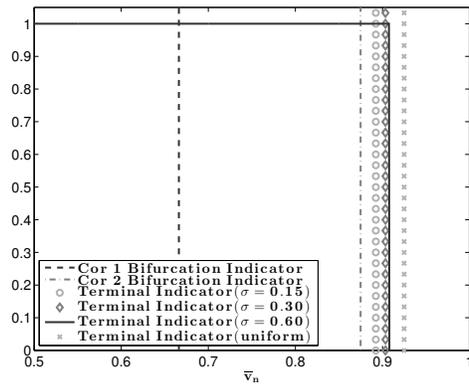
$$\underline{b} = \max \left(\arg \max_b (\underline{v}_n - b) F_1(b)^{m_1} \right). \quad (20)$$

To understand (20), recall that the first m_1 bidders bid their true value when their type is below \underline{b} . The expression in the parenthesis says that bidder n with type \underline{v}_n must best respond to this bidding behavior.

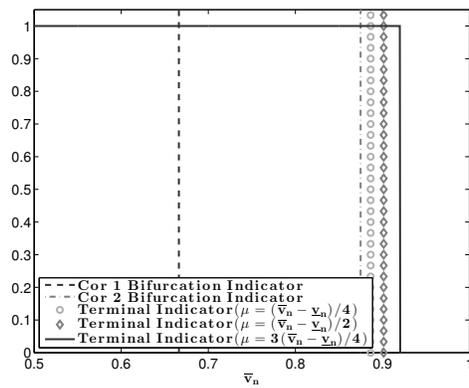
In summary, if $m_1, m_n \geq 2$, then the initial condition is that $\varphi_i(\underline{v}) = \underline{v}$, $i = 1, n$. Here, the smallest serious bid is $\underline{b} = \underline{v}$. The same is true if $m_n = 1$ and $\underline{v}_1 > \underline{v}_n$. If $m_n = 1$ and $\underline{v}_n > \underline{v}_1$, however, the initial condition is that $\varphi_n(\underline{v}_n) = \underline{b}$ and $\varphi_1(\underline{b}) = \underline{b}$, where \underline{b} is determined by (20).



(a) Uniform Example

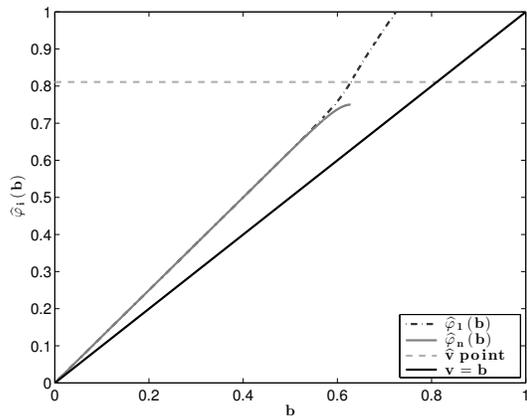


(b) Truncated Normal (σ_n)

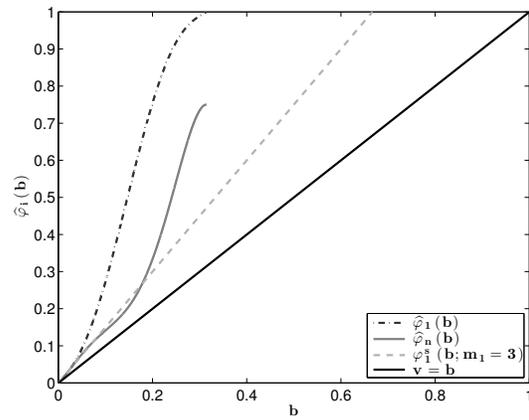


(c) Truncated Normal (μ_n)

Figure 4: Predictions of Diagnostic Test for Bid-separation

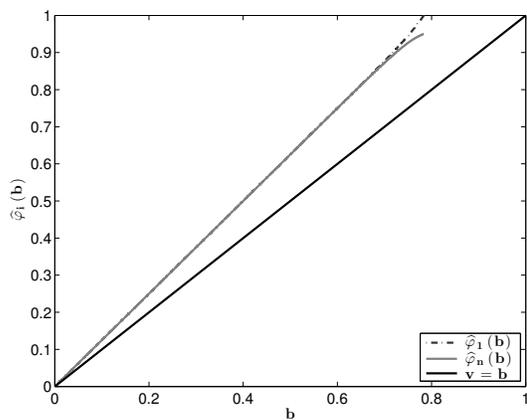


(a) Bid-separation (Correct)

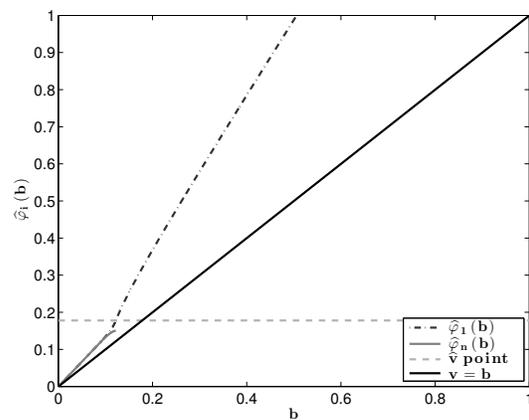


(b) Common Bid Support (Incorrect)

Figure 5: EXAMPLE 1: Imposing Bid-separation vs. Common Bid Support

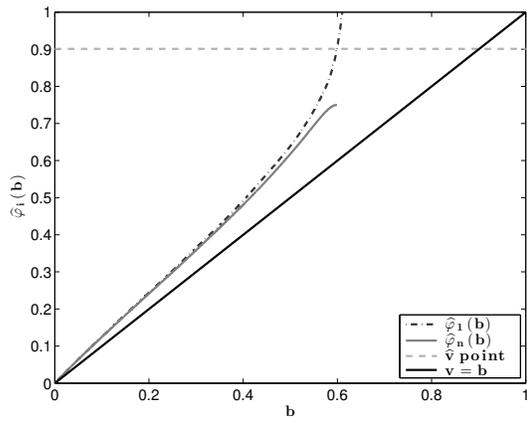


(a) Inverse Bid Functions: EXAMPLE 2

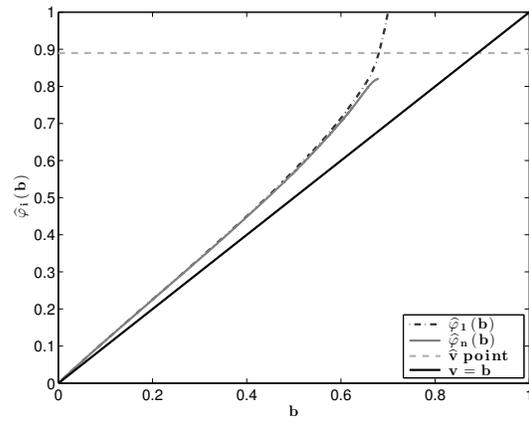


(b) Inverse Bid Functions: EXAMPLE 3

Figure 6: EXAMPLES 2 & 3: Inverse Bid Functions



(a) Inverse Bid Functions: EXAMPLE 4



(b) Inverse Bid Functions: EXAMPLE 5

Figure 7: EXAMPLES 4 & 5: Inverse Bid Functions